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On irregular prime power divisors of the Bernoulli numbers

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Abstract

Let B_n ($n = 0, 1, 2, \ldots$) denote the usual n-th Bernoulli number. Let l be a positive even integer where $l = 12$ or $l \geq 16$. It is well known that the numerator of the reduced quotient $|B_l/l|$ is a product of powers of irregular primes. Let (p, l) be an irregular pair with $B_l/l \not\equiv B_{l+p-1}/(l + p - 1) \pmod{p^2}$. We show that for every $r \geq 1$ the congruence $B_{m_r}/m_r \equiv 0 \pmod{p^r}$ has a unique solution m_r where $m_r \equiv l \pmod{p-1}$ and $l \leq m_r < (p-1)p^{r-1}$. The sequence $(m_r)_{r \geq 1}$ defines a p-adic integer $\chi_{(p,l)}$ which is a zero of a certain p-adic zeta function $\zeta_{p,l}$ originally defined by T. Kubota and H. W. Leopoldt. We show some properties of these functions and give some applications. Subsequently we give several computations of the (truncated) p-adic expansion of $\chi_{(p,l)}$ for irregular pairs (p, l) with p below 1000.

Keywords: Bernoulli number, Riemann zeta function, p-adic zeta function, Kummer congruences, irregular prime power, irregular pair of higher order

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1 Introduction

The classical Bernoulli numbers B_n are defined by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad |z| < 2\pi.$$

These numbers are rational where $B_n = 0$ for odd $n > 1$ and $(-1)^{\frac{n}{2}+1} B_n > 0$ for even $n > 0$. The Bernoulli numbers are connected with the Riemann zeta function $\zeta(s)$ on the positive real axis by Euler's formula

$$\zeta(n) = -\frac{1}{2} \frac{(2\pi i)^n}{n!} B_n, \quad n \in \mathbb{N},\ 2 \mid n; \tag{1.1}$$

the functional equation of $\zeta(s)$ leads to

$$\zeta(1-n) = -\frac{B_n}{n} \quad \text{for} \quad n \in \mathbb{N},\ n \geq 2. \tag{1.2}$$

Let, as usual, φ denote Euler's totient function. The Kummer congruences state for $n, m, p, r \in \mathbb{N}$, n, m even, p prime with $p - 1 \nmid n$ that

$$(1 - p^{n-1}) \frac{B_n}{n} \equiv (1 - p^{m-1}) \frac{B_m}{m} \pmod{p^r} \tag{1.3}$$

when $n \equiv m \pmod{\varphi(p^r)}$; see [9, Thm. 5, p. 239].

In 1850 E. E. Kummer [14] introduced two classes of odd primes, later called regular and irregular. An odd prime p is called *regular* if p does not divide the class number of the cyclotomic field $\mathbb{Q}(\mu_p)$ where μ_p is the set of p-th roots of unity; otherwise *irregular*. Kummer proved that Fermat's last theorem (FLT) is true if the exponent is a regular prime. Kummer also gave an equivalent definition of irregularity concerning Bernoulli numbers. We recall the usual definition from [9, pp. 233–234].

Definition 1.1 Let p be an odd prime. The pair (p, l) is called an *irregular pair* if p divides the numerator of B_l where l is even and $2 \leq l \leq p-3$. The *index of irregularity* of p is defined to be

$$i(p) := \#\{(p, l) \text{ is an irregular pair} : \ l = 2, 4, \ldots, p-3\}.$$

The prime p is called a *regular* prime if $i(p) = 0$, otherwise an *irregular* prime.

We introduce the following notations for rational numbers. If q is rational then we use the representation $q = N/D$ where $(N, D) = 1$ and $D > 0$. We define $\operatorname{denom}(q) = D$ resp. $\operatorname{numer}(q) = N$ for the denominator resp. the numerator of q. The notation $m \mid q$ where m is a positive integer means that $m \mid \operatorname{numer}(q)$; we shall also write $q \equiv 0 \pmod{m}$ in this case.

For now, let n be an even positive integer. An elementary property of the Bernoulli numbers, independently discovered by T. Clausen [6] and K. G. C. von Staudt [20] in 1840, is the following. The structure of the denominator of B_n is given by

$$B_n + \sum_{p-1 \mid n} \frac{1}{p} \in \mathbb{Z} \qquad \text{which implies} \qquad \operatorname{denom}(B_n) = \prod_{p-1 \mid n} p. \tag{1.4}$$

A further result, often associated with the name of J. C. Adams, see [9, Prop. 15.2.4, p. 238] and Section 8 below, is that B_n/n is a p-integer for all primes p with $p-1 \nmid n$. Therefore

$$\prod_{p-1 \nmid n} p^{\operatorname{ord}_p n} \tag{1.5}$$

divides $\operatorname{numer}(B_n)$; since this factor is cancelled in the numerator of B_n/n, we shall call it the *trivial factor* of B_n. By $|B_n| > 2(n/2\pi e)^n$, see [9, Eq. (8), p. 232], and Table A.1, one can easily show that the numerator of $|B_n/n|$ equals 1 for $2 \leq n \leq 10$ and $n = 14$. Otherwise this numerator is a product of powers of irregular primes; this is a consequence of the Kummer congruences. The determination of irregular primes resp. irregular pairs is still a difficult task, see [2]. One can easily show that infinitely many irregular primes exist; for a short proof see Carlitz [4]. Unfortunately, the more difficult question of whether infinitely many regular primes exist is still open. However, calculations in [2] show that about 60% of all primes less than 12 million are regular which agree with an expected distribution proposed by Siegel [18].

The values of B_n and B_n/n for $n \leq 20$ are given in Table A.1, irregular pairs with $p < 1000$ in Table A.3. For brevity we write $\widehat{B}(n) = B_n/n$; these are called the *divided* Bernoulli numbers. Throughout this paper all indices concerning Bernoulli numbers are even and p denotes an odd prime.

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2 Preliminaries

The definition of irregular pairs can be extended to irregular prime powers as was first proposed by the author [11, Section 2.5].

Definition 2.1 Let p be an odd prime and n, l be positive integers. A pair (p,l) is called an *irregular pair of order* n if $p^n \mid \widehat{B}(l)$ where l is even and $2 \leq l < \varphi(p^n)$. Define

$$\Psi_n^{\mathrm{irr}} := \{(p,l) : p^n \mid \widehat{B}(l),\, p \text{ is an odd prime},\, 2 \leq l < \varphi(p^n),\, 2 \mid l\}$$

as the set of irregular pairs of order n. For a prime p the *index* of irregular pairs of order n is defined by

$$i_n(p) := \#\{(p,l) : (p,l) \in \Psi_n^{\mathrm{irr}}\}.$$

Define the map

$$\lambda_n : \Psi_{n+1}^{\mathrm{irr}} \to \Psi_n^{\mathrm{irr}}, \quad (p,l) \mapsto (p, l \bmod \varphi(p^n))$$

where $x \bmod y$ gives the least nonnegative residue x' with $0 \leq x' < y$ for positive integers x and y. Let $(p,l_n) \in \Psi_n^{\mathrm{irr}}$ and $(p,l_m) \in \Psi_m^{\mathrm{irr}}$ be irregular pairs of order n resp. m where $n > m \geq 1$. We say that (p,l_n) is *related* to (p,l_m) if $l_n \equiv l_m \pmod{\varphi(p^m)}$ holds. Note that "related" is not a symmetric relation.

Remark 2.2 This definition includes the older Definition 1.1 in the case $n = 1$. Therefore one has $i_1(p) = i(p)$. Let $(p,l) \in \Psi_n^{\mathrm{irr}}$ with $n \geq 1$. The Kummer congruences (1.3) imply that $p^n \mid \widehat{B}(l + \nu\varphi(p^n))$ for all $\nu \in \mathbb{N}_0$. Conversely, if $p^n \mid \widehat{B}(m)$ for some even integer m, then there exists an irregular pair $(p,l) \in \Psi_n^{\mathrm{irr}}$ where $l \equiv m \pmod{\varphi(p^n)}$ with $l \leq m$ holds.

The map λ_n is well defined by the properties mentioned above. Let $(p,l_n) \in \Psi_n^{\mathrm{irr}}$ and $(p,l_m) \in \Psi_m^{\mathrm{irr}}$ where $n > m \geq 1$ and (p,l_n) is related to (p,l_m). By applying the maps $\lambda_{n-1}, \lambda_{n-2}, \ldots, \lambda_m$ one derives a chain of related irregular pairs of descending order

$$(p,l_n) \in \Psi_n^{\mathrm{irr}},\ (p,l_{n-1}) \in \Psi_{n-1}^{\mathrm{irr}},\ (p,l_{n-2}) \in \Psi_{n-2}^{\mathrm{irr}},\ \ldots,\ (p,l_m) \in \Psi_m^{\mathrm{irr}} \tag{2.1}$$

where

$$l_n \geq l_{n-1} \geq l_{n-2} \geq \ldots \geq l_m.$$

Definition 2.3 For $(p,l) \in \Psi_n^{\mathrm{irr}}$, $n \geq 1$ define

$$\Delta_{(p,l)} \equiv p^{-n}\left(\widehat{B}(l + \varphi(p^n)) - \widehat{B}(l)\right) \pmod{p}$$

with $0 \leq \Delta_{(p,l)} < p$. When $\Delta_{(p,l)} = 0$ we call $\Delta_{(p,l)}$ *singular*. For an irregular prime p set

$$\Delta(p) := \begin{cases} 1, & \Delta_p \neq 0 \\ 0, & \Delta_p = 0 \end{cases}$$

with

$$\Delta_p = \prod_{\nu=1}^{i(p)} \Delta_{(p,l_\nu)}, \quad (p, l_\nu) \in \Psi_1^{\mathrm{irr}}.$$

Then $\Delta(p) = 1$ if and only if all $\Delta_{(p,l_\nu)}$ are nonsingular.

We need a generalized form of the Kummer congruences which allows us to obtain most of the later results, see Carlitz [3, Thm. 3, p. 425], especially Fresnel [7, Cor. 6, p. 319].

Theorem 2.4 (Carlitz) *Let $k, m, n, p, r, \omega \in \mathbb{N}$, m even, p prime with $p - 1 \nmid m$, and $\omega = k\varphi(p^n)$. Then*

$$\sum_{\nu=0}^{r} \binom{r}{\nu} (-1)^\nu (1 - p^{m+\nu\omega-1}) \widehat{B}(m + \nu\omega) \equiv 0 \pmod{p^{nr}}. \tag{2.2}$$

Here we need a special version without Euler factors which p-adically shows the periodic behavior of the divided Bernoulli numbers.

Corollary 2.5 *Let $(p, l) \in \Psi_n^{\mathrm{irr}}$, $n \geq 1$. Let $k, m, r, \omega \in \mathbb{N}$, $r > 1$, and $\omega = k\varphi(p^n)$. For $m = l + j\varphi(p^n)$ with $j \geq 0$ we have*

$$\sum_{\nu=0}^{r} \binom{r}{\nu} (-1)^\nu \, p^{-n} \widehat{B}(m + \nu\omega) \equiv 0 \pmod{(p^{m-1}, p^{n(r-1)})}.$$

Proof. Since $(p, l) \in \Psi_n^{\mathrm{irr}}$ we know that $p^n \mid \widehat{B}(l + j\varphi(p^n))$ for all $j \geq 0$. Hence, we can reduce Congruence (2.2) to $\pmod{p^{n(r-1)}}$ when multiplying it by p^{-n}. One easily sees that all Euler factors in the sum of (2.2) vanish $\pmod{p^{m-1}}$. $\square$

Proposition 2.7 below shows how to find irregular pairs of higher order. Beginning from an irregular pair of order n, we can characterize related irregular pairs of order $n + 1$ if they exist. First we need a lemma.

Lemma 2.6 *Let n be a positive integer and p be an odd prime. Let $(\alpha_\nu)_{\nu \geq 0}$ be a sequence of p-integers $\alpha_\nu \in \mathbb{Q}$ for all $\nu \in \mathbb{N}_0$. If one has*

$$\alpha_\nu - 2\alpha_{\nu+1} + \alpha_{\nu+2} \equiv 0 \pmod{p^n} \tag{2.3}$$

then the sequence is equidistant $\pmod{p^n}$. For $\alpha_0 \not\equiv \alpha_1 \pmod{p}$ the elements α_0 up to α_{p^n-1} run through all residues $\pmod{p^n}$. Then exactly one element $\alpha_s \equiv 0 \pmod{p^n}$ exists with $0 \leq s < p^n$ where $s \equiv -\alpha_0(\alpha_1 - \alpha_0)^{-1} \pmod{p^n}$.

PROOF. We rewrite Congruence (2.3) to

$$\alpha_\nu - \alpha_{\nu+1} \equiv \alpha_{\nu+1} - \alpha_{\nu+2} \pmod{p^n}, \quad \nu \in \mathbb{N}_0.$$

One easily observes by induction on ν that all elements α_ν are equidistant $\pmod{p^n}$. Let $\delta \equiv \alpha_1 - \alpha_0 \pmod{p^n}$, then we obtain

$$\alpha_\nu \equiv \alpha_0 + \delta\nu \pmod{p^n}.$$

The case $\alpha_0 \not\equiv \alpha_1 \pmod{p}$ provides that $\alpha_0 + \delta\nu$ resp. α_ν runs through all residues $\pmod{p^n}$ for $0 \le \nu < p^n$, since δ is invertible $\pmod{p^n}$. Then exactly one element α_s exists with $0 \equiv \alpha_s \equiv \alpha_0 + \delta s \pmod{p^n}$ and $0 \le s < p^n$. This finally gives $s \equiv -\alpha_0\,\delta^{-1} \equiv -\alpha_0(\alpha_1 - \alpha_0)^{-1} \pmod{p^n}$. $\square$

Proposition 2.7 *Let $(p, l) \in \Psi_n^{\mathrm{irr}}$, $n \ge 1$. Let the sequence $(\alpha_j)_{j \ge 0}$ satisfy*

$$\alpha_j \equiv p^{-n}\widehat{B}(l + j\varphi(p^n)) \pmod{p}.$$

For $\Delta_{(p,l)} \equiv \alpha_1 - \alpha_0 \pmod{p}$ where $0 \le \Delta_{(p,l)} < p$ there exist three cases:

(1) If $\Delta_{(p,l)} = 0$ and $\alpha_0 \not\equiv 0 \pmod{p}$, then there are no related irregular pairs of order $n+1$ and higher,

(2) If $\Delta_{(p,l)} = 0$ and $\alpha_0 \equiv 0 \pmod{p}$, then all $(p, l + \nu\varphi(p^n)) \in \Psi_{n+1}^{\mathrm{irr}}$ are related irregular pairs of order $n+1$ for $\nu = 0, \ldots, p-1$,

(3) If $\Delta_{(p,l)} \ne 0$, then exactly one related irregular pair of order $n+1$ exists. One has $(p, l + s\varphi(p^n)) \in \Psi_{n+1}^{\mathrm{irr}}$ with $0 \le s < p$ where $s \equiv -\alpha_0\Delta_{(p,l)}^{-1} \pmod{p}$.

PROOF. Let $j \ge 0$. Using Corollary 2.5 with $r = 2$, $\omega = \varphi(p^n)$, and $m = l + j\varphi(p^n) \ge 2$, we get

$$\sum_{\nu=0}^{2} \binom{2}{\nu} (-1)^\nu\, p^{-n}\widehat{B}(m + \nu\omega) \equiv 0 \pmod{p}$$

which is

$$\alpha_j - 2\alpha_{j+1} + \alpha_{j+2} \equiv 0 \pmod{p}.$$

This satisfies the conditions of Lemma 2.6. We obtain three cases:

Case (1): We have $\alpha_0 \equiv \alpha_1 \pmod{p}$ and $\alpha_0 \not\equiv 0 \pmod{p}$. One observes that $\alpha_j \not\equiv 0 \pmod{p}$ resp. $p^{n+1} \nmid \widehat{B}(l + j\varphi(p^n))$ for all $j \ge 0$. Therefore, there cannot exist related irregular pairs of order $n+1$. Also there cannot exist related irregular pairs of order $r > n+1$, otherwise we would get a contradiction to (2.1).

Case (2): We have $\alpha_0 \equiv \alpha_1 \pmod{p}$ and $\alpha_0 \equiv 0 \pmod{p}$. This yields $\alpha_j \equiv 0 \pmod{p}$ resp. $p^{n+1} \mid \widehat{B}(l + j\varphi(p^n))$ for all $j \ge 0$. Hence, p related irregular pairs of order $n+1$ exist where $(p, l + \nu\varphi(p^n)) \in \Psi_{n+1}^{\mathrm{irr}}$ for $\nu = 0, \ldots, p-1$.

Case (3): We have $\alpha_0 \not\equiv \alpha_1 \pmod{p}$. Lemma 2.6 provides exactly one element $\alpha_s \equiv 0 \pmod{p}$ with the desired properties. Hence, $(p, l + s\varphi(p^n))$ is the only related irregular pair of order $n+1$. $\square$

Remark 2.8 Vandiver [19] describes the result of the previous proposition for the case $n = 1$ and only for the first irregular primes 37, 59, and 67. For these primes Pollaczek [16] has calculated the indices s of the now called irregular pair of order two, but case $p = 67$ with $s = 2$ is incorrect, see column s_2 of Table A.3. This error was already noticed by Johnson [10] who has also determined all irregular pairs (p, l') of order two with p below 8000. Wagstaff [21] has extended calculations of irregular pairs, indices s, and associated cyclotomic invariants up to $p < 125\,000$. He also checked that FLT is true for all such exponents p in that range. Finally, Buhler, Crandall, Ernvall, Metsänkylä, and Shokrollahi [2] have extended calculations of irregular pairs and associated cyclotomic invariants up to $p < 12\,000\,000$. For all these irregular pairs (p, l) in that range $\Delta_{(p,l)} \neq 0$ is always valid which ensures that each time there is only one related irregular pair (p, l') of order two. Hence $i_2(p) = i(p)$ for these irregular primes p. One has to notice that always $(p, l) \neq (p, l')$. So far, no irregular pair (p, l) has been found with $p^2 \mid \widehat{B}(l)$.

Using Proposition 2.7 one can successively find irregular pairs of higher order. We can easily extend the result starting from an irregular pair $(p, l) \in \Psi_n^{\mathrm{irr}}$ and requiring that $l > n$ to obtain a related irregular pair $(p, l') \in \Psi_{2n}^{\mathrm{irr}}$.

Proposition 2.9 *Let $(p, l) \in \Psi_n^{\mathrm{irr}}$, $n \geq 1$. Suppose that $l > n$. Let the sequence $(\alpha_j)_{j \geq 0}$ satisfy*

$$\alpha_j \equiv p^{-n} \widehat{B}(l + j\varphi(p^n)) \pmod{p^n}.$$

If $\Delta_{(p,l)} \neq 0$, then there is exactly one related irregular pair of order $2n$

$$(p, l + s\varphi(p^n)) \in \Psi_{2n}^{\mathrm{irr}}$$

with $0 \leq s < p^n$ where $s \equiv -\alpha_0(\alpha_1 - \alpha_0)^{-1} \pmod{p^n}$. Correspondingly, there also exists exactly one related irregular pair of order $n + \nu$

$$(p, l + s_\nu \varphi(p^n)) \in \Psi_{n+\nu}^{\mathrm{irr}}$$

for each $\nu = 1, \ldots, n-1$ with $0 \leq s_\nu < p^\nu$ where $s_\nu \equiv s \pmod{p^\nu}$.

Proof. Let $j \geq 0$. Using Corollary 2.5 again with $r = 2$, $\omega = \varphi(p^n)$, and $m = l + j\varphi(p^n) > n$ yields

$$\alpha_j - 2\alpha_{j+1} + \alpha_{j+2} \equiv 0 \pmod{p^n}.$$

If $\Delta_{(p,l)} \neq 0$, then Lemma 2.6 provides exactly one element $\alpha_s \equiv 0 \pmod{p^n}$ with $0 \leq s < p^n$ where $s \equiv -\alpha_0(\alpha_1 - \alpha_0)^{-1} \pmod{p^n}$. Therefore, $(p, l + s\varphi(p^n))$ is the only related irregular pair of order $2n$. Similarly, regarding the congruences above $\pmod{p^\nu}$ instead of $\pmod{p^n}$ for $\nu = 1, \ldots, n-1$ yields the proposed properties. $\square$

Finally, we start from an irregular pair $(p, l) \in \Psi_n^{\mathrm{irr}}$ where we have to suppose that $l > (r-1)n$ with $r \geq 2$ to obtain a related irregular pair $(p, l') \in \Psi_{rn}^{\mathrm{irr}}$.

Proposition 2.10 *Let $(p,l) \in \Psi_n^{\mathrm{irr}}$, $n \geq 1$. Let $r > 1$ be an integer and suppose that $l > (r-1)n$. Let the sequence $(\alpha_j)_{j \geq 0}$ satisfy*

$$\alpha_j \equiv p^{-n} \widehat{B}(l + j\varphi(p^n)) \pmod{p^{(r-1)n}}.$$

Then this sequence satisfies for all $j \geq 0$:

$$\sum_{\nu=0}^{r} \binom{r}{\nu} (-1)^\nu \alpha_{\nu+j} \equiv 0 \pmod{p^{(r-1)n}}.$$

The elements α_0 up to α_{r-1} induce the entire sequence $(\alpha_j)_{j \geq 0}$. Elements with $\alpha_s \equiv 0 \pmod{p^{(r-1)n}}$ where $0 \leq s < p^{(r-1)n}$ provide related irregular pairs of order rn with $(p, l + s\varphi(p^n)) \in \Psi_{rn}^{\mathrm{irr}}$. If $\Delta_{(p,l)} \neq 0$ and the elements α_0 up to α_{r-1} are equidistant $\pmod{p^{(r-1)n}}$, then there is exactly one related irregular pair of order rn

$$(p, l + s\varphi(p^n)) \in \Psi_{rn}^{\mathrm{irr}}$$

with $0 \leq s < p^{(r-1)n}$ where $s \equiv -\alpha_0(\alpha_1 - \alpha_0)^{-1} \pmod{p^{(r-1)n}}$. Correspondingly, there exists exactly one related irregular pair of order $n+k$

$$(p, l + s_k\varphi(p^n)) \in \Psi_{n+k}^{\mathrm{irr}}$$

for each $k = 1, \ldots, (r-1)n - 1$ with $0 \leq s_k < p^k$ where $s_k \equiv s \pmod{p^k}$.

Proof. Let $j \geq 0$. Clearly, by Corollary 2.5 with $r > 1$, $\omega = \varphi(p^n)$, and $m = l + j\varphi(p^n) > (r-1)n$, we have

$$\sum_{\nu=0}^{r} \binom{r}{\nu} (-1)^\nu \alpha_{\nu+j} \equiv 0 \pmod{p^{(r-1)n}}.$$

This induces the whole sequence $(\alpha_j)_{j \geq 0}$ by

$$\alpha_{r+j} \equiv (-1)^{r+1} \sum_{\nu=0}^{r-1} \binom{r}{\nu} (-1)^\nu \alpha_{\nu+j} \pmod{p^{(r-1)n}}. \tag{2.4}$$

Among all elements α_s with $0 \leq s < p^{(r-1)n}$, an element $\alpha_s \equiv 0 \pmod{p^{(r-1)n}}$ provides a related irregular pair $(p, l + s\varphi(p^n)) \in \Psi_{rn}^{\mathrm{irr}}$ of order rn.

Now we assume that $\Delta_{(p,l)} \neq 0$ and the first elements α_0 up to α_{r-1} are equidistant $\pmod{p^{(r-1)n}}$. We show that this property transfers to the entire sequence. Let γ, δ be integers. It is easily seen for $r > 1$ that

$$\sum_{\nu=0}^{r} \binom{r}{\nu} (-1)^\nu (\gamma + \delta\nu) = 0. \tag{2.5}$$

Consider $\gamma \equiv \alpha_0 \pmod{p^{(r-1)n}}$ and $\delta \equiv \alpha_1 - \alpha_0 \pmod{p^{(r-1)n}}$ where $p \nmid \delta$ by assumption. Combining (2.4) and (2.5) yields

$$(-1)^{r+1} \sum_{\nu=0}^{r-1} \binom{r}{\nu} (-1)^\nu (\alpha_0 + \delta\nu) \equiv \alpha_0 + \delta r \equiv \alpha_r \pmod{p^{(r-1)n}} \tag{2.6}$$

which shows inductively that all successive elements α_j with $j \geq r$ are equidistant $\pmod{p^{(r-1)n}}$. Since δ is invertible $\pmod{p^{(r-1)n}}$, exactly one solution exists with $0 \equiv \alpha_s \equiv \alpha_0 + \delta s \pmod{p^{(r-1)n}}$ where $s \equiv -\alpha_0(\alpha_1 - \alpha_0)^{-1} \pmod{p^{(r-1)n}}$. Using similar arguments, Congruences (2.4) and (2.6) are also valid $\pmod{p^k}$ for $k = 1, \ldots, (r-1)n-1$ which provides for each k a unique solution s_k with $0 \leq s_k < p^k$ where $s_k \equiv s \pmod{p^k}$ holds. $\square$

In [11, pp. 125–130] several examples and calculations are given which use the previous propositions. These results are reprinted in Table A.4 and following. It will turn out later that calculations can be further simplified. Because of the rapid growth of indices, it is useful to write indices of irregular pairs of higher order p-adically.

Definition 2.11 Let $(p, l) \in \Psi_n^{\mathrm{irr}}$, $n \geq 1$. We write

$$(p, s_1, s_2, \ldots, s_n) \in \widehat{\Psi}_n^{\mathrm{irr}} \quad \text{where} \quad l = \sum_{\nu=1}^{n} s_\nu \, \varphi(p^{\nu-1})$$

for the p-adic notation of (p, l) with $0 \leq s_\nu < p$ for $\nu = 1, \ldots, n$ and $2 \leq s_1 \leq p-3$, $2 \mid s_1$. The corresponding set is denoted as $\widehat{\Psi}_n^{\mathrm{irr}}$, the map corresponding to λ_n is given by

$$\widehat{\lambda}_n : \widehat{\Psi}_{n+1}^{\mathrm{irr}} \to \widehat{\Psi}_n^{\mathrm{irr}}, \quad (p, s_1, s_2, \ldots, s_n, s_{n+1}) \mapsto (p, s_1, s_2, \ldots, s_n).$$

The pair (p, l) and the element $(p, s_1, s_2, \ldots, s_n)$ are called *associated*.

Remark 2.12 The definition of $\widehat{\Psi}_n^{\mathrm{irr}}$ means that we have $\Psi_1^{\mathrm{irr}} = \widehat{\Psi}_1^{\mathrm{irr}}$ for $n = 1$. For $n \geq 2$ we can define a map $\Psi_n^{\mathrm{irr}} \to \widehat{\Psi}_n^{\mathrm{irr}}$, $(p, l) \mapsto (p, s_1, \ldots, s_n)$ where the s_k are uniquely determined by the p-adic representation

$$l = s_1 + (p-1)\hat{s}, \quad \hat{s} = \sum_{\nu=0}^{n-2} s_{\nu+2} \, p^\nu, \quad 0 \leq s_{\nu+2} < p$$

and by $s_1 \equiv l \pmod{p-1}$ with $2 \leq s_1 \leq p-3$. If $s_k = 0$ with $k \geq 2$ then there is an irregular pair (p, l_k) of order k with

$$(p, l_k) \in \Psi_k^{\mathrm{irr}} \quad \text{and} \quad (p, l_k) \in \Psi_{k-1}^{\mathrm{irr}}.$$

Note that $(p, s_1, s_2, \ldots, s_n)$ is also called an *irregular pair* with $(s_1, s_2, \ldots, s_n)$ as the second parameter given p-adically.

3 Main results

Theorem 3.1 *Let $(p, l_1) \in \Psi_1^{\mathrm{irr}}$. If $\Delta_{(p,l_1)} \neq 0$ then for each $n > 1$ there exists exactly one related irregular pair of order n. There is a unique sequence $(l_n)_{n \geq 1}$ resp. $(s_n)_{n \geq 1}$ with*

$$(p, l_n) \in \Psi_n^{\mathrm{irr}} \quad \text{resp.} \quad (p, s_1, \ldots, s_n) \in \widehat{\Psi}_n^{\mathrm{irr}}$$

and

$$l_1 \leq l_2 \leq l_3 \leq \ldots, \quad \lim_{n \to \infty} l_n = \infty.$$

Moreover one has

$$\Delta_{(p,l_1)} = \Delta_{(p,l_2)} = \Delta_{(p,l_3)} = \dots .$$

If $\Delta(p) = 1$ *then*

$$i(p) = i_2(p) = i_3(p) = \dots .$$

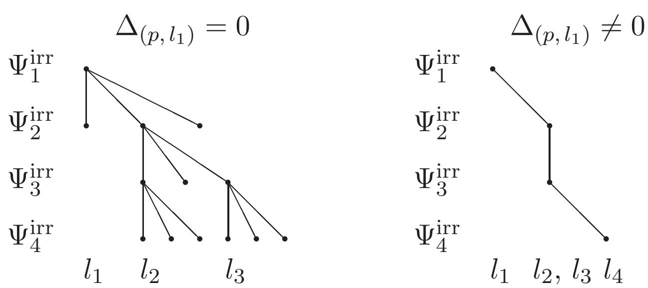
Theorem 3.2 *Let* $(p, l_n) \in \Psi_n^{\mathrm{irr}}$, $n \geq 1$ *with* $\Delta_{(p,l_n)} = 0$. *Then there are two cases:*

(1) $(p, l_n) \notin \Psi_{n+1}^{\mathrm{irr}}$: *There are no related irregular pairs of order* $n+1$ *and higher,*

(2) $(p, l_n) \in \Psi_{n+1}^{\mathrm{irr}}$: *There exist* p *related irregular pairs of order* $n+1$ *where* $(p, l_{n+1,j}) \in \Psi_{n+1}^{\mathrm{irr}}$ *with* $\Delta_{(p,\, l_{n+1,j})} = 0$ *and* $l_{n+1,j} = l_n + j\varphi(p^n)$ *for* $j = 0, \dots, p-1$.

The property of $\Delta_{(p,\,l)}$, whether $\Delta_{(p,\,l)}$ vanishes or not, is passed on to all related irregular pairs of higher order. The case of a singular $\Delta_{(p,\,l)}$ would possibly imply a strange behavior without any regularity. By calculation in [2] up to $p < 12\,000\,000$, no such $\Delta_{(p,\,l)}$ was found. The following diagram illustrates both cases.

Diagram 3.3



Here a vertical line indicates that $(p, l_n) \in \Psi_n^{\mathrm{irr}} \cap \Psi_{n+1}^{\mathrm{irr}}$ happens. On the left side we then have p related irregular pairs of order $n+1$ which are represented by branches. In this case the corresponding Bernoulli number $\widehat{B}(l_n)$ *decides* whether there exist further branches or they stop. Instead of n the order of the p-power must be at least $n+1$. This also means that an associated irregular pair $(p, s_1, \dots, s_{n+1}) \in \widehat{\Psi}_{n+1}^{\mathrm{irr}}$ must have a zero $s_{n+1} = 0$ each time. In contrast to, the right side shows that in the case of $\Delta_{(p,l_1)} \neq 0$ there is only one related irregular pair of each higher order. If $\Delta(p) = 1$ then higher powers p^ν are equally distributed among the numerators of $\widehat{B}(n)$. For each irregular pair considered, there exists exactly one index $n_{k,\nu}$ with $n_{k,\nu} = n_{0,\nu} + k\varphi(p^\nu)$, $k \in \mathbb{N}_0$ in the disjoint intervals

$$(k\,\varphi(p^\nu), (k+1)\,\varphi(p^\nu))$$

for which $p^\nu \mid \widehat{B}(n_{k,\nu})$ is valid.

In [11, pp. 128–130] irregular pairs of order 10 were calculated for all irregular primes $p < 1000$. These results are reprinted in Table A.3. In this table only one irregular pair has a zero in its p-adic notation:

$$(157, 62, 40, 145, 67, 29, 69, 0, 87, 89, 21) \in \widehat{\Psi}_{10}^{\mathrm{irr}}.$$

Hence, one has with a *relatively small* index that

$$(157, 6\,557\,686\,520\,486) \in \Psi_6^{\mathrm{irr}} \cap \Psi_7^{\mathrm{irr}}.$$

It seems that these zeros can be viewed as exceptional; see also Table A.2. It would be of interest to investigate in which regions such indices may occur. This could explain why no irregular pair $(p, l) \in \Psi_1^{\mathrm{irr}} \cap \Psi_2^{\mathrm{irr}}$ has yet been found, because these regions are beyond present calculations. Here we have index 12 000 000 in [2] against index 6 557 686 520 486. Because of the rare occurrence of zeros one can expect that $(p, l) \in \Psi_1^{\mathrm{irr}} \cap \Psi_2^{\mathrm{irr}}$ resp. $p^2 \mid \widehat{B}(l)$ will not happen often.

Another phenomenon is the occurrence of huge irregular prime factors. Wagstaff [22] has completely factored the numerators of the Bernoulli numbers B_n with index up to $n = 152$. Most of these irregular prime factors are large numbers, the greatest factors have 30 up to 100 digits.

Finally, summarizing all facts together, the property $\Delta(p) = 1$ can be viewed as a structural property of the Bernoulli numbers. This leads us to the following conjecture named by the author Δ-Conjecture.

Conjecture 3.4 (Δ-Conjecture) *For all irregular primes p the following properties, which are equivalent, hold:*

(1) $\Delta_{(p,l)}$ *is nonsingular for all irregular pairs* $(p, l) \in \Psi_1^{\mathrm{irr}}$,

(2) $\Delta(p) = 1$,

(3) $i(p) = i_2(p) = i_3(p) = \ldots.$

Assuming the Δ-Conjecture one can also prove the existence of infinitely many irregular primes using only information about the numerators of divided Bernoulli numbers, see [11, Satz 2.8.2, p. 87]. Now we give the proofs of the theorems above.

Proposition 3.5 *Let $(p, l_n) \in \Psi_n^{\mathrm{irr}}$, $n \geq 1$ with $\Delta_{(p,l_n)} \neq 0$. Then there is exactly one related irregular pair $(p, l_{n+1}) \in \Psi_{n+1}^{\mathrm{irr}}$ with $\Delta_{(p,l_n)} = \Delta_{(p,l_{n+1})}$.*

Proof. We write $\Delta_n = \Delta_{(p,l_n)}$. Note that $l_n > 2$ and $p > 3$. Define the sequence $(\alpha_j)_{j \geq 0}$ by

$$\alpha_j \equiv p^{-n} \widehat{B}(l_n + j\varphi(p^n)) \pmod{p^2}.$$

Using Corollary 2.5 with $r = 3$, $\omega = \varphi(p^n)$, and $m = l_n + j\varphi(p^n) > 2$, we have for $n(r-1) \geq 2$ that

$$\alpha_j - 3\alpha_{j+1} + 3\alpha_{j+2} - \alpha_{j+3} \equiv 0 \pmod{p^2}.$$

Taking differences with $\beta_j = \alpha_{j+1} - \alpha_j$ yields

$$\beta_j - 2\beta_{j+1} + \beta_{j+2} \equiv 0 \pmod{p^2}.$$

By Lemma 2.6 the sequence $(\beta_j)_{j \geq 0}$ is equidistant $(\mathrm{mod}\ p^2)$. Proposition 2.7 shows that the sequence $(\alpha_j)_{j \geq 0}$ is equidistant $(\mathrm{mod}\ p)$. By definition we have $\beta_j \equiv \Delta_n \pmod{p}$. Therefore, we can choose suitable $\gamma, \delta \in \mathbb{Z}$ so that

$$\alpha_{j+1} - \alpha_j \equiv \beta_j \equiv \Delta_n + p\,(\gamma + j\delta) \pmod{p^2}.$$

This yields

$$\begin{aligned}\alpha_j &\equiv \alpha_0 + \sum_{\nu=0}^{j-1} (\Delta_n + p\,(\gamma + \nu\delta)) \\ &\equiv \alpha_0 + j\Delta_n + jp\gamma + \binom{j}{2} p\delta \pmod{p^2}. \end{aligned} \tag{3.1}$$

From Proposition 2.7 we have

$$s \equiv -\alpha_0\, \Delta_n^{-1} \pmod{p}, \quad 0 \le s < p. \tag{3.2}$$

With $l_{n+1} = l_n + s\varphi(p^n)$ we obtain the unique related irregular pair $(p, l_{n+1}) \in \Psi_{n+1}^{\mathrm{irr}}$ of order $n+1$. As a consequence of Lemma 2.6, we observe that

$$\alpha_{s+jp} \equiv 0 \pmod{p}.$$

Thus we obtain a sequence $(\alpha'_j)_{j\ge 0}$ defined by

$$\alpha'_j \equiv \alpha_{s+jp}/p \equiv p^{-(n+1)} \widehat{B}(l_{n+1} + j\varphi(p^{n+1})) \pmod{p}$$

which we can use to determine related irregular pairs of order $n+2$ using Proposition 2.7. By definition we have

$$\Delta_{n+1} \equiv \alpha'_1 - \alpha'_0 \pmod{p}.$$

It follows from (3.1) that

$$\begin{aligned} p\,\Delta_{n+1} &\equiv p\,(\alpha'_1 - \alpha'_0) \equiv \alpha_{s+p} - \alpha_s \\ &\equiv p\,\Delta_n + p^2\,\gamma + p\,\delta \left(\binom{s+p}{2} - \binom{s}{2} \right) \\ &\equiv p\,\Delta_n \qquad\qquad \pmod{p^2}, \end{aligned}$$

since

$$\binom{s+p}{2} - \binom{s}{2} = \frac{1}{2}\, p\,(p + 2s - 1).$$

Finally, we obtain the proposed equation $\Delta_{n+1} = \Delta_n$. $\square$

Proof of Theorem 3.1. Using Proposition 3.5 with induction on n provides

$$\Delta_{(p,l_1)} = \Delta_{(p,l_2)} = \Delta_{(p,l_3)} = \dots$$

with exactly one related irregular pair of order n

$$(p, l_n) \in \Psi_n^{\mathrm{irr}} \quad \text{resp.} \quad (p, s_1, \dots, s_n) \in \widehat{\Psi}_n^{\mathrm{irr}}.$$

The latter pair is given by Definition 2.11. Proposition 2.7 shows for each step n that

$$l_{n+1} = l_n + s_n\, \varphi(p^n), \quad 0 \le s_n < p \tag{3.3}$$

with a suitable integer s_n. This ensures that $l_1 \leq l_2 \leq l_3 \leq \ldots$ as an increasing sequence $(l_j)_{j\geq 1}$. Clearly, this sequence is not eventually constant, because $p^n \mid \widehat{B}(l_n)$ with $0 < |\widehat{B}(l_n)| < \infty$. Therefore $\lim_{n\to\infty} l_n = \infty$.

Starting with $(p, l_1) \in \Psi_1^{\mathrm{irr}}$ this provides a unique sequence $(l_j)_{j\geq 1}$. If we have another irregular pair $(p, l_1') \in \Psi_1^{\mathrm{irr}}$ with $\Delta_{(p,l_1')} \neq 0$ and $(p, l_1) \neq (p, l_1')$ then

$$l_j' \neq l_k \quad \text{for all } j, k \in \mathbb{N},$$

because $l_1' \not\equiv l_1 \pmod{\varphi(p)}$ and $l_1' \equiv l_j' \pmod{\varphi(p)}$ resp. $l_1 \equiv l_k \pmod{\varphi(p)}$ by (3.3).

If $\Delta(p) = 1$ then for each of the $i(p)$ irregular pairs $(p, l_{1,\nu})$, $\nu = 1, \ldots, i(p)$ there exists exactly one related irregular pair of higher order. Finally, it follows that $i(p) = i_2(p) = i_3(p) = \ldots$ and so on. $\square$

Proof of Theorem 3.2. Clearly, the (non-) existence of related irregular pairs in the case (1) resp. (2) is given by Proposition 2.7 case (1) resp. (2). Hence, we only have to show the remaining part of case (2). In this case we have p related irregular pairs $(p, l_{n+1,\nu}) = (p, l_n + \nu\varphi(p^n)) \in \Psi_{n+1}^{\mathrm{irr}}$ of order $n+1$ with $\nu = 0, \ldots, p-1$. Although $\Delta_{(p,l_n)} = 0$ we can use Proposition 3.5 by modifying essential steps. We then have

$$\alpha_j \equiv \beta_j \equiv 0 \pmod{p}.$$

Congruence (3.2) must be replaced by

$$s = 0, \ldots, p-1 \tag{3.2'}$$

since $\Delta_{(p,l_n)} = 0$ yields p values of s. It follows that

$$\Delta_{(p,l_n)} = \Delta_{(p,l_{n+1,\nu})} = 0 \quad \text{for} \quad \nu = 0, \ldots, p-1. \qquad \square$$

4 A p-adic view

Let $\mathbb{Z}_p$ be the ring of p-adic integers and $\mathbb{Q}_p$ be the field of p-adic numbers. The ultrametric absolute value $|\ |_p$ is defined by $|x|_p = p^{-\operatorname{ord}_p x}$ on $\mathbb{Q}_p$. Define $|\ |_\infty$ as the standard norm on $\mathbb{Q}$. For $n \geq 1$ we define $\psi_n : \mathbb{Z}_p \to \mathbb{Z}$ giving the projection of $\mathbb{Z}_p$ onto the set $[0, p^n) \cap \mathbb{Z}$ so that for $x \in \mathbb{Z}_p$ we have $x - \psi_n(x) \in p^n\mathbb{Z}_p$ where $0 \leq \psi_n(x) < p^n$. We denote $\mathbb{P}$ as the set of the rational primes. Now, we shall use (1.2) to reformulate our results. Let $(p, l) \in \Psi_1^{\mathrm{irr}}$ then

$$\Delta_{(p,\,l)} \equiv \frac{\zeta(1-l) \,-\, \zeta(1-(l+p-1))}{p} \pmod{p}$$

with $0 \leq \Delta_{(p,\,l)} < p$.

Theorem 4.1 *Let $(p, l) \in \Psi_1^{\mathrm{irr}}$ with $\Delta_{(p,\,l)} \neq 0$. We define the sequence $(l_n)_{n\geq 1}$ recursively by $l_1 = l$ and, for $n \geq 1$, by*

$$l_{n+1} \,=\, l_n + \varphi(p)\,\psi_n\!\left(\frac{\zeta(1-l_n)}{p\,\Delta_{(p,\,l)}}\right) \,=\, l_n + \varphi(p^n)\,\psi_1\!\left(\frac{\zeta(1-l_n)}{p^n\,\Delta_{(p,\,l)}}\right).$$

Then we have $\zeta(1-l_n) \in p^n \mathbb{Z}_p$ *and consequently*

$$\lim_{n\to\infty} |\zeta(1-l_n)|_p = 0 \quad \textit{with} \quad l_n \to \infty.$$

PROOF. We rewrite our results using $\zeta(1-l_n) = -\widehat{B}(l_n)$. Theorem 3.1 provides one, and only one, sequence $(l_n)_{n\geq 1}$ with $l_1 = l$ and $(p, l_n) \in \Psi_n^{\mathrm{irr}}$ with $l_n \to \infty$. This implies the p-adic convergence $\zeta(1-l_n) \to 0$. Additionally, from Proposition 2.7 we have

$$l_{n+1} = l_n + s\,\varphi(p^n)$$

for each step where

$$s \equiv -p^{-n} \widehat{B}(l_n)\, \Delta_{(p,\,l)}^{-1} \pmod{p}, \quad 0 \leq s < p.$$

Rewriting the last congruence yields

$$s = \psi_1\left(\frac{\zeta(1-l_n)}{p^n\, \Delta_{(p,\,l)}}\right).$$

The rest follows by $\psi_n(a\,p^{n-1}) = p^{n-1}\,\psi_1(a)$ for $a \in \mathbb{Z}_p$. $\square$

These results can be also applied to the so-called p-adic zeta functions which were originally defined by T. Kubota and H. W. Leopoldt [13] in 1964; for a detailed theory see Koblitz [12, Chapter II]. We introduce some definitions.

Definition 4.2 Let p be a prime. For $n \geq 1$ define

$$\zeta_p(1-n) \;:=\; (1-p^{n-1})\,\zeta(1-n) \;=\; -(1-p^{n-1})\,\widehat{B}(n).$$

Define the *p-adic zeta function* for $p \geq 5$ and a fixed $s_1 \in \{2, 4, \ldots, p-3\}$ by

$$\zeta_{p,\,s_1} :\; \mathbb{Z}_p \to \mathbb{Z}_p, \qquad \zeta_{p,\,s_1}(s) := \lim_{t_\nu \to s} \zeta_p\big(1-(s_1+(p-1)t_\nu)\big)$$

resp. for $p \geq 2$ and $s_1 = 0$ by

$$\zeta_{p,\,0} :\; \mathbb{Z}_p \setminus \{0\} \to \mathbb{Q}_p, \qquad \zeta_{p,\,0}(s) := \lim_{t_\nu \to s} \zeta_p\big(1-(p-1)t_\nu\big)$$

for p-adic integers s by taking any sequence $(t_\nu)_{\nu\geq 1}$ of nonnegative integers resp. positive integers in the case $s_1 = 0$ which p-adically converges to s.

Remark 4.3 Let $p \geq 2$ and $s_1 \geq 0$. The p-adic zeta function $\zeta_{p,\,s_1}(s)$ interpolates the zeta function $\zeta_p(1-n)$ at nonnegative integer values s by

$$\zeta_{p,\,s_1}(s) = \zeta_p(1-n)$$

where $n \equiv s_1 \pmod{p-1}$ and $n = s_1 + (p-1)s$.

Let $p \geq 5$ and $s_1 \neq 0$. The Kummer congruences (1.3) state for $r \geq 1$ that

$$\zeta_{p,\,s_1}(s) \equiv \zeta_{p,\,s_1}(s') \pmod{p^r}$$

when $s \equiv s' \pmod{p^{r-1}}$ for nonnegative integers s and s'. Since $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, the p-adic zeta function $\zeta_{p,\,s_1}$ extends uniquely, by means of the interpolation property and the Kummer congruences, to a continuous function on $\mathbb{Z}_p$; see [12, Thm. 8, p. 46].

Definition 4.4 Let $(p, l) \in \Psi_1^{\mathrm{irr}}$ with $\Delta_{(p,l)} \neq 0$. Define a *characteristic* p-adic integer which contains all information about irregular pairs of higher order by

$$\chi_{(p,l)} = \sum_{\nu \geq 0} s_{\nu+2}\, p^\nu \ \in \mathbb{Z}_p$$

where $(s_\nu)_{\nu \geq 1}$ is the sequence defined by Theorem 3.1 and $l = s_1$.

Lemma 4.5 *Let $(p, l) \in \Psi_1^{\mathrm{irr}}$ with $\Delta_{(p,l)} \neq 0$. Let $r = \mathrm{ord}_p\, \widehat{B}(l)$ and $(p, s_1, \ldots, s_{r+1}) \in \widehat{\Psi}_{r+1}^{\mathrm{irr}}$ be the related irregular pair of order $r+1$. Then*

$$s_{r+1}\, \Delta_{(p,l)} \equiv -p^{-r}\, \widehat{B}(l) \pmod{p}$$

with $s_1 = l$, $s_\nu = 0$ for $\nu = 2, \ldots, r$, $s_{r+1} \neq 0$. If $r = 1$ then $\chi_{(p,l)} \in \mathbb{Z}_p^*$, otherwise $\chi_{(p,l)} \in p^{r-1}\mathbb{Z}_p$ for $r \geq 2$.

Proof. Since $r = \mathrm{ord}_p\, \widehat{B}(l) \geq 1$, we have $(p, l) \in \Psi_\nu^{\mathrm{irr}}$ for all $\nu = 1, \ldots, r$. Then Proposition 2.7 and Theorem 3.1 provide

$$s \equiv -p^{-r}\widehat{B}(l)\, \Delta_{(p,l)}^{-1} \pmod{p}, \qquad (p, l + s\,\varphi(p^r)) \in \Psi_{r+1}^{\mathrm{irr}}$$

with $0 \leq s < p$. We have $s \neq 0$ since $\mathrm{ord}_p\, (p^{-r}\widehat{B}(l)) = 0$. Let $(p, s_1, \ldots, s_{r+1}) \in \widehat{\Psi}_{r+1}^{\mathrm{irr}}$ be the related irregular pair of order $r+1$. Then we see that $s_1 = l$, $s_{r+1} = s \neq 0$, and $s_\nu = 0$ for $\nu = 2, \ldots, r$. By Definition 4.4 we have

$$\chi_{(p,l)} = s_2 + s_3\, p + \ldots + s_{r+1}\, p^{r-1} + \ldots .$$

Case $r = 1$ yields $s_2 \neq 0$ and $\chi_{(p,l)} \in \mathbb{Z}_p^*$, otherwise case $r \geq 2$ implies that $\mathrm{ord}_p\, \chi_{(p,l)} = r - 1$ and $\chi_{(p,l)} \in p^{r-1}\mathbb{Z}_p$. $\square$

Theorem 4.6 *Let $(p, l) \in \Psi_1^{\mathrm{irr}}$ with $\Delta_{(p,l)} \neq 0$. The p-adic zeta function $\zeta_{p,l}(s)$ has a unique zero at $s = \chi_{(p,l)}$.*

Proof. From Theorem 3.1 we have a sequence $(l_n)_{n \geq 1}$ with $l = l_1$. In view of Definition 4.2, Theorem 4.1 also states that

$$\lim_{n \to \infty} |\zeta_p(1 - l_n)|_p = 0 \quad \text{with} \quad l_n \to \infty.$$

We can transfer this result to the p-adic zeta function $\zeta_{p,l}$ by the interpolation property. We see that p-adically

$$\lim_{n \to \infty} l_n = l + (p-1)\chi_{(p,l)}.$$

Since the function $\zeta_{p,l}$ is continuous, the p-adic integer $\chi_{(p,l)}$ is a zero of $\zeta_{p,l}$. We shall show that this zero is unique. Assume that $\zeta_{p,l}(\xi) = 0$ with some $\xi \in \mathbb{Z}_p$. We can use the arguments given above in the opposite direction. Since $\zeta_{p,l}$ is continuous, there exists a sequence $(l'_n)_{n \geq 1}$ of positive integers with

$$\lim_{n \to \infty} l'_n = l + (p-1)\xi \quad \text{and} \quad \lim_{n \to \infty} |\zeta_p(1 - l_n)|_p = 0.$$

We can choose a subsequence $(l''_n)_{n\geq 1}$ of $(l'_n)_{n\geq 1}$ such that $\zeta_p(1-l''_n) \in p^n\mathbb{Z}_p$. By use of the Kummer congruences, we construct the sequence $(\tilde{l}_n)_{n\geq 1}$ where $\tilde{l}_n \equiv l''_n \pmod{\varphi(p^n)}$ with $l \leq \tilde{l}_n < \varphi(p^n)$. Now we have $(p, \tilde{l}_n) \in \Psi_n^{\mathrm{irr}}$ for all $n \geq 1$. Since $l = l_1 = \tilde{l}_1$, Theorem 3.1 shows that $(l_n)_{n\geq 1} = (\tilde{l}_n)_{n\geq 1}$. This implies that $\xi = \chi_{(p,\,l)}$. □

Remark 4.7 Let $(p, l) \in \Psi_1^{\mathrm{irr}}$ with $\Delta_{(p,\,l)} \neq 0$. The related irregular pair $(p, s_1, \ldots, s_r)$ $\in \widehat{\Psi}_r^{\mathrm{irr}}$ is a p-adic approximation of the zero $\chi_{(p,\,l)}$ of the p-adic zeta function $\zeta_{p,\,l}$. For the first irregular primes 37, 59, and 67 elements of $\widehat{\Psi}_{100}^{\mathrm{irr}}$ were calculated in [11, pp. 127–128]. These results are reprinted in Table A.2. By Lemma 4.5 the statement $p^2 \nmid B_l$ is equivalent to the fact that $\chi_{(p,\,l)}$ is a unit in $\mathbb{Z}_p$.

From now on, we shall assume the Δ-Conjecture. We shall see that the zeros $\chi_{(p,\,l)}$ play an important role in the representation of the Riemann zeta function at odd negative integer arguments.

Theorem 4.8 *Let n be an even positive integer. Under the assumption of the Δ-Conjecture we have*

$$\zeta(1-n) = (-1)^{\frac{n}{2}} \prod_{p-1\mid n} \frac{|n|_p}{p} \prod_{\substack{(p,l)\in\Psi_1^{\mathrm{irr}} \\ l\equiv n \pmod{p-1}}} \frac{p}{|\chi_{(p,\,l)} - \frac{n-l}{p-1}|_p}.$$

Proof. Since both products above have only positive terms, the sign follows by (1.2). The first product describes the denominator of $\zeta(1-n)$ which is a consequence of (1.4) and (1.5). We have to show that the second product describes the unsigned numerator of $\zeta(1-n)$ which only consists of powers of irregular primes. Let p be an irregular prime divisor of $\zeta(1-n)$. From Remark 2.2 we have

$$\operatorname{ord}_p \widehat{B}(n) = r \quad \Longrightarrow \quad n \equiv l_r \pmod{\varphi(p^r)} \quad \text{with} \quad (p, l_r) \in \Psi_r^{\mathrm{irr}}.$$

The irregular pair (p, l_r) of order r is related to some irregular pair $(p, l) \in \Psi_1^{\mathrm{irr}}$ with $\Delta_{(p,\,l)} \neq 0$ where $l \equiv l_r \equiv n \pmod{\varphi(p)}$. We also have by Definition 4.4 that

$$n \equiv l_r \equiv l + (p-1)\,\chi_{(p,\,l)} \pmod{(p-1)p^{r-1}\mathbb{Z}_p}$$

and equally by reduction that

$$\frac{n-l}{p-1} \equiv \chi_{(p,\,l)} \pmod{p^{r-1}\mathbb{Z}_p}.$$

The last congruence is not valid $\pmod{p^r\mathbb{Z}_p}$ by construction. Therefore

$$\left|\chi_{(p,\,l)} - \frac{n-l}{p-1}\right|_p = p^{-(r-1)}$$

which provides, with an additional factor p, the desired p-power in the second product. Considering all irregular primes p which can appear, the second product equals the numerator of $\zeta(1-n)$ without sign. □

With some technical definitions we can combine both products of the theorem above. This yields a more accessible representation of the Riemann zeta function by means of p-adic analysis.

Theorem 4.9 *Define $\Psi_0 = \Psi_1^{\mathrm{irr}} \cup (\mathbb{P} \times \{0\})$ and set $\chi_{(p,0)} = 0$ for all $p \in \mathbb{P}$. Define $\rho(l) = 1 - 2\,\mathrm{sign}(l) = \pm 1$ for $l \geq 0$. Let n be an even positive integer. Under the assumption of the Δ-Conjecture we have*

$$\zeta(1-n) = (-1)^{\frac{n}{2}} \prod_{\substack{(p,l) \in \Psi_0 \\ l \equiv n \pmod{p-1}}} \left(\frac{|\chi_{(p,l)} - \frac{n-l}{p-1}|_p}{p} \right)^{\rho(l)} .$$

PROOF. We only have to consider case $l = 0$. Then we have $p - 1 \mid n$, $\rho(0) = 1$, and $|\chi_{(p,0)} - \frac{n}{p-1}|_p = |n|_p$. The other case $l > 0$ is already covered by Theorem 4.8. $\square$

We shall give an interpretation of this formula above in Remark 4.17 by means of p-adic zeta functions. This generalization shows the significance to prove the Δ-Conjecture at all.

Theorem 4.10 *Let $(p,l) \in \Psi_1^{\mathrm{irr}}$ with $\Delta_{(p,l)} \neq 0$. Let $s, t \in \mathbb{Z}_p$. Then a strong version of the Kummer congruences holds that*

$$|\zeta_{p,l}(s) - \zeta_{p,l}(t)|_p = |p\,(s-t)|_p.$$

Moreover

$$\frac{\zeta_{p,l}(s) - \zeta_{p,l}(t)}{p\,(s-t)} \equiv -\Delta_{(p,l)} \pmod{p\mathbb{Z}_p} \quad \text{for} \quad s \neq t$$

and

$$\zeta'_{p,l}(s) \equiv -p\,\Delta_{(p,l)} \pmod{p^2\mathbb{Z}_p}.$$

Thus, $\Delta_{(p,l)}$ is closely associated with the p-adic zeta function $\zeta_{p,l}$ in the nonsingular case $\Delta_{(p,l)} \neq 0$. We will prove this theorem later.

Corollary 4.11 *Let $(p,l) \in \Psi_1^{\mathrm{irr}}$ with $\Delta_{(p,l)} \neq 0$. The p-adic zeta function $\zeta_{p,l}(s)$ has a simple zero at $s = \chi_{(p,l)}$. Moreover, for $s \in \mathbb{Z}_p$,*

$$\zeta_{p,l}(s) = p\,(s - \chi_{(p,l)})\,\zeta^{\star}_{p,l}(s)$$

where $\zeta^{\star}_{p,l}(s)$ is a continuous function on $\mathbb{Z}_p$ with $\zeta^{\star}_{p,l}(s) \equiv -\Delta_{(p,l)} \pmod{p\mathbb{Z}_p}$. Consequently

$$|\zeta_{p,l}(s)|_p = |p\,(s - \chi_{(p,l)})|_p.$$

PROOF. Theorem 4.6 shows that $\zeta_{p,l}(s)$ has a unique zero at $s = \chi_{(p,l)}$. We can use Theorem 4.10 to define

$$\zeta^{\star}_{p,l}(s) = \begin{cases} \dfrac{\zeta_{p,l}(s)}{p(s - \chi_{(p,l)})}, & s \neq \chi_{(p,l)}, \\[2ex] \dfrac{\zeta'_{p,l}(s)}{p}, & s = \chi_{(p,l)}. \end{cases}$$

Theorem 4.10 implies that $\zeta^{\star}_{p,\,l}(s) \equiv -\Delta_{(p,\,l)} \pmod{p\mathbb{Z}_p}$ for all $s \in \mathbb{Z}_p$. Hence $\zeta^{\star}_{p,\,l}(s)$ has no zeros and consequently $\zeta_{p,\,l}(s)$ has a simple zero at $s = \chi_{(p,\,l)}$. Since $\zeta_{p,\,l}(s)$ is continuous on $\mathbb{Z}_p$ and $\zeta'_{p,\,l}(s)$ exists at $s = \chi_{(p,\,l)}$, the function $\zeta^{\star}_{p,\,l}(s)$ is also continuous on $\mathbb{Z}_p$. Finally we obtain $\zeta_{p,\,l}(s) = p\,(s - \chi_{(p,\,l)})\,\zeta^{\star}_{p,\,l}(s)$ and $|\zeta_{p,\,l}(s)|_p = |p\,(s-\chi_{(p,\,l)})|_p\,|\zeta^{\star}_{p,\,l}(s)|_p = |p\,(s-\chi_{(p,\,l)})|_p$. □

Definition 4.12 Let p be a prime. Define

$$\zeta^{\star}_{p,\,0} : \mathbb{Z}_p \to \mathbb{Z}_p, \qquad \zeta^{\star}_{p,\,0}(s) := -\lim_{t_\nu \to s}(1 - p^{t_\nu(p-1)-1})\frac{pB_{t_\nu(p-1)}}{p-1}$$

for p-adic integers s by taking any sequence $(t_\nu)_{\nu \geq 1}$ of nonnegative integers which p-adically converges to s.

Proposition 4.13 *Let p be a prime. The function $\zeta^{\star}_{p,\,0}(s)$ is continuous on $\mathbb{Z}_p \setminus \{0\}$ and satisfies the following properties:*

$$\zeta^{\star}_{p,\,0}(0) = -1, \quad \zeta^{\star}_{p,\,0}(s) = ps\,\zeta_{p,\,0}(s), \quad s \in \mathbb{Z}_p \setminus \{0\}$$

and

$$\lim_{s \to 0} |ps\,\zeta_{p,\,0}(s) - \zeta^{\star}_{p,\,0}(0)|_p < 1.$$

Moreover $\zeta^{\star}_{p,\,0}(s) \equiv -1 \pmod{p\mathbb{Z}_p}$ for $s \in \mathbb{Z}_p$ in case $p > 2$ and $s \in p\mathbb{Z}_p$ in case $p = 2$. Additionally, if $p = 2$ then $\zeta^{\star}_{p,\,0}(s) = 0$ for $s \equiv 1 \pmod{p\mathbb{Z}_p}$.

Proof. From Definition 4.2 and Definition 4.12 we have for $s \in \mathbb{Z}_p \setminus \{0\}$ and any sequence $(t_\nu)_{\nu \geq 1}$ of positive integers which p-adically converges to s that

$$\begin{aligned} ps\,\zeta_{p,\,0}(s) &= -\lim_{t_\nu \to s}(1 - p^{t_\nu(p-1)-1})ps\frac{B_{t_\nu(p-1)}}{t_\nu(p-1)} \\ &= -\lim_{t_\nu \to s}(1 - p^{t_\nu(p-1)-1})\frac{pB_{t_\nu(p-1)}}{p-1} \quad = \quad \zeta^{\star}_{p,\,0}(s). \end{aligned}$$

Moreover we have

$$\zeta^{\star}_{p,\,0}(0) = -(1-p^{-1})\,pB_0/(p-1) = -1.$$

By Clausen-von Staudt (1.4) we obtain $\zeta^{\star}_{p,\,0}(s) \equiv -1 \pmod{p\mathbb{Z}_p}$ for $s \in \mathbb{Z}_p$ in case $p > 2$ and $s \in p\mathbb{Z}_p$ in case $p = 2$. Additionally we have $\zeta^{\star}_{2,\,0}(s) = 0$ for $s \in 1 + 2\mathbb{Z}_2$, since $\zeta^{\star}_{2,\,0}(1) = -(1-2^0)\,2B_1 = 0$ and $B_n = 0$ for all odd integers $n > 1$. We use the fact that $\zeta_{p,\,0} : \mathbb{Z}_p \to \mathbb{Q}_p$ is a continuous function on $\mathbb{Z}_p \setminus \{0\}$; see [12, Thm. 8, p. 46]. Hence $\zeta^{\star}_{p,\,0}(s) = ps\,\zeta_{p,\,0}(s)$ is also continuous on $\mathbb{Z}_p \setminus \{0\}$. It remains to show that

$$\lim_{s \to 0} |ps\,\zeta_{p,\,0}(s) - \zeta^{\star}_{p,\,0}(0)|_p < 1.$$

We can choose a zero sequence $(s_\nu)_{\nu \geq 1}$ where its elements are arbitrary close to 0, say $0 < |s_\nu|_p < p^{-r}$ with some positive integer r. Then we deduce that

$$ps_\nu\,\zeta_{p,\,0}(s_\nu) - \zeta^{\star}_{p,\,0}(0) \equiv \zeta^{\star}_{p,\,0}(s_\nu) + 1 \equiv 0 \pmod{p\mathbb{Z}_p}$$

for all $\nu \geq 1$. This implies, together with the continuity of $ps\,\zeta_{p,\,0}(s)$, the estimate above. □

Corollary 4.14 *Let p be a prime. The p-adic zeta function $\zeta_{p,0}(s)$ has a simple pole at $s = 0$. Moreover, for $s \in \mathbb{Z}_p \setminus \{0\}$,*

$$\zeta_{p,0}(s) = \zeta^{\star}_{p,0}(s)/(ps).$$

Consequently

$$|\zeta_{p,0}(s)|_p = |ps|_p^{-1}$$

for $s \in \mathbb{Z}_p \setminus \{0\}$ in case $p > 2$ and $s \in p\mathbb{Z}_p \setminus \{0\}$ in case $p = 2$.

Proof. By Proposition 4.13 we can write $\zeta_{p,0}(s) = \zeta^{\star}_{p,0}(s)/(ps)$ for $s \in \mathbb{Z}_p \setminus \{0\}$. Moreover $\lim_{s\to 0} ps\,\zeta_{p,0}(s) = \xi$ with $|\xi - \zeta^{\star}_{p,0}(0)|_p < 1$. This implies that the limit $\lim_{s\to 0} \zeta_{p,0}(s)$ does not exist and that $\zeta_{p,0}(s)$ has a simple pole at $s = 0$. We have $|\zeta^{\star}_{p,0}(s)|_p = 1$ for $s \in \mathbb{Z}_p \setminus \{0\}$ in case $p > 2$ and $s \in p\mathbb{Z}_p \setminus \{0\}$ in case $p = 2$. In these cases we then obtain $|\zeta_{p,0}(s)|_p = |ps|_p^{-1}$. □

Remark 4.15 One can even show further that $\zeta^{\star}_{p,0}(s)$ is continuous on $\mathbb{Z}_p$ where $\lim_{s\to 0} \zeta^{\star}_{p,0}(s) = \zeta^{\star}_{p,0}(0)$; moreover $\zeta^{\star}_{p,0}(s)$ satisfies the Kummer congruences. This can be derived, e.g., by means of p-adic integration, see Koblitz [12, pp. 42–46], or by using certain congruences of Carlitz [5].

Theorem 4.16 *Define $\Psi_0 = \Psi_1^{\mathrm{irr}} \cup (\mathbb{P} \times \{0\})$ and set $\chi_{(p,0)} = 0$ for all $p \in \mathbb{P}$. Define $s_{p,l}(n) = (n-l)/(p-1)$. Let n be an even positive integer. Then*

$$|\zeta(1-n)|_\infty = \prod_{\substack{p\in\mathbb{P} \\ l\equiv n \pmod{p-1}}} |\zeta_{p,l}(s_{p,l}(n))|_p^{-1} = \prod_{\substack{(p,l)\in\Psi_0 \\ l\equiv n \pmod{p-1}}} |\zeta_{p,l}(s_{p,l}(n))|_p^{-1}.$$

Under the assumption of the Δ-Conjecture we have

$$|\zeta(1-n)|_\infty = \prod_{\substack{(p,l)\in\Psi_1^{\mathrm{irr}} \\ l\equiv n \pmod{p-1}}} |p(s_{p,l}(n) - \chi_{(p,l)})|_p^{-1} \Big/ \prod_{p-1|n} |p(s_{p,0}(n) - \chi_{(p,0)})|_p^{-1}.$$

Proof. Since n is an even positive integer, the product formula states that

$$\prod_{p\,\in\,\mathbb{P}\,\cup\,\{\infty\}} |\zeta(1-n)|_p = 1.$$

By Definition 4.2 we have $|\zeta(1-n)|_p = |\zeta_{p,l}(s)|_p$ where $l \equiv n \pmod{p-1}$ with $0 \le l < p-1$ and $s = s_{p,l}(n) = (n-l)/(p-1)$. Thus

$$|\zeta(1-n)|_\infty^{-1} = \prod_{p\,\in\,\mathbb{P}} |\zeta(1-n)|_p = \prod_{\substack{p\in\mathbb{P} \\ l\equiv n \pmod{p-1}}} |\zeta_{p,l}(s_{p,l}(n))|_p\,.$$

We have $|\zeta_{p,l}(s)|_p = 1$ for $s \in \mathbb{Z}_p$ if $(p,l) \notin \Psi_1^{\mathrm{irr}}$ and $l \neq 0$. Hence

$$\prod_{\substack{p\in\mathbb{P} \\ l\equiv n \pmod{p-1}}} |\zeta_{p,l}(s_{p,l}(n))|_p = \prod_{\substack{(p,l)\in\Psi_0 \\ l\equiv n \pmod{p-1}}} |\zeta_{p,l}(s_{p,l}(n))|_p\,.$$

Next we assume the Δ-Conjecture. By Corollary 4.11 we then have for $(p,l) \in \Psi_1^{\mathrm{irr}}$ that $|\zeta_{p,l}(s)|_p = |p\,(s - \chi_{(p,l)})|_p$ for $s \in \mathbb{Z}_p$. Without any assumption, Corollary 4.14 shows that $|\zeta_{p,0}(s)|_p = |ps|_p^{-1} = |p(s - \chi_{(p,0)})|_p^{-1}$ for $s \in \mathbb{Z}_p \setminus \{0\}$ in case $p > 2$ and $s \in p\mathbb{Z}_p \setminus \{0\}$ in case $p = 2$. Since n is even and $s_{2,0}(n) = n$, we finally obtain

$$\prod_{\substack{(p,l)\in\Psi_0 \\ l\equiv n \pmod{p-1}}} |\zeta_{p,l}(s_{p,l}(n))|_p = \prod_{\substack{(p,l)\in\Psi_1^{\mathrm{irr}} \\ l\equiv n \pmod{p-1}}} |p(s_{p,l}(n) - \chi_{(p,l)})|_p \prod_{p-1|n} |p(s_{p,0}(n) - \chi_{(p,0)})|_p^{-1}.$$

□

Remark 4.17 Assuming the Δ-Conjecture, the numerator of $|\zeta(1-n)|_\infty$ with even $n > 0$ is essentially described by simple zeros $\chi_{(p,l)}$ of p-adic zeta functions $\zeta_{p,l}(s)$ where $l \equiv n \pmod{p-1}$ and $(p,l) \in \Psi_1^{\mathrm{irr}}$. More precisely, by the variable substitution $s = (n-l)/(p-1)$, the term

$$|p(s - \chi_{(p,l)})|_p^{-1}$$

is equal to p^r for some suitable $r > 0$ which is the p power factor in the numerator of $|\zeta(1-n)|_\infty$. The term above is induced by

$$|\zeta_{p,l}(s)|_p = |p(s - \chi_{(p,l)})|_p = p^{-1}|(s - \chi_{(p,l)})|_p\,.$$

Since $\chi_{(p,l)}$ is a simple zero of $\zeta_{p,l}(s)$, $|\zeta_{p,l}(s)|_p$ is mainly determined by a linear factor which one can also interpret as a distance between s and $\chi_{(p,l)}$.

Similar arguments can be applied to the denominator of $|\zeta(1-n)|_\infty$. Without any assumption, the denominator of $|\zeta(1-n)|_\infty$ is essentially described by simple poles $\chi_{(p,0)}$ of p-adic zeta functions $\zeta_{p,0}(s)$ where $p-1 \mid n$. Again, the term

$$|p(s - \chi_{(p,0)})|_p^{-1}$$

is equal to p^r for some suitable $r > 0$ which is the p power factor in the denominator of $|\zeta(1-n)|_\infty$. This term is induced by

$$|\zeta_{p,0}(s)|_p = |p(s - \chi_{(p,0)})|_p^{-1} = p\,|(s - \chi_{(p,0)})|_p^{-1}$$

where $\zeta_{p,0}(s)$ has a simple pole at $s = \chi_{(p,0)}$.

Now we shall make some preparations to give later a proof of Theorem 4.10.

Definition 4.18 Define the linear finite-difference operator $\mathcal{D}$ and its powers by

$$\mathcal{D}^r f(s) = \sum_{\nu=0}^{r} \binom{r}{\nu} (-1)^\nu f(s+\nu)$$

for $r \geq 0$ and any function $f : \mathbb{Z}_p \to \mathbb{Z}_p$. The series

$$f(s) = \sum_{\nu \geq 0} a_\nu \binom{s}{\nu}$$

with coefficients $a_\nu \in \mathbb{Z}_p$ where $|a_\nu|_p \to 0$ is called a *Mahler series* which defines a continuous function $f : \mathbb{Z}_p \to \mathbb{Z}_p$.

The following theorem of Mahler shows that the converse also holds, see [17, Thm. 1, p. 173]. Note that the sign $(-1)^\nu$ depends on the definition of $\mathcal{D}$.

Theorem 4.19 (Mahler) *Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be a continuous function. Then f has a Mahler series*

$$f(s) = \sum_{\nu \geq 0} a_\nu \binom{s}{\nu}$$

where the coefficients a_ν are given by $a_\nu = (-1)^\nu \, \mathcal{D}^\nu f(0)$ and $|a_\nu|_p \to 0$ holds.

Definition 4.20 Let $(p, l) \in \Psi_1^{\text{irr}}$ with $\Delta_{(p,l)} \neq 0$. For $n \geq 1$ define the p-adic zeta function of order n by

$$\zeta_{p,\,l,\,n}(s) = p^{-n} \, \zeta_{p,\,l} \left(\psi_{n-1}(\chi_{(p,\,l)}) + p^{n-1} s \right), \quad s \in \mathbb{Z}_p$$

which for $s = 0$ corresponds to the related irregular pair of order n.

Proposition 4.21 *Let $(p, l) \in \Psi_1^{\text{irr}}$ with $\Delta_{(p,l)} \neq 0$. For positive integers n, r we have*

$$\mathcal{D}^{r+1} \zeta_{p,\,l,\,n}(s) \equiv 0 \pmod{p^{nr} \mathbb{Z}_p}, \quad s \in \mathbb{Z}_p \tag{4.1}$$

and

$$\zeta_{p,\,l,\,n}(s) \equiv \zeta_{p,\,l,\,n}(t) \pmod{p^r \mathbb{Z}_p}, \quad s, t \in \mathbb{Z}_p \tag{4.2}$$

when $s \equiv t \pmod{p^r \mathbb{Z}_p}$.

Proof. First assume that $s \in \mathbb{N}_0$. In analogy to Corollary 2.5, we have to modify Theorem 2.4 in a similar way, since $\zeta_{p,\,l,\,n}(0)$ corresponds to the related irregular pair of order n. Rewriting (2.2) in the case $k = 1$ gives (4.1). Since $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, we can extend (4.1) to values $s \in \mathbb{Z}_p$ by means of the interpolation property of $\zeta_{p,\,l}$ resp. $\zeta_{p,\,l,\,n}$. By the same arguments the Kummer congruences, given in Remark 4.3, can be extended to values in $\mathbb{Z}_p$. Let $s \equiv t \pmod{p^r \mathbb{Z}_p}$ and write $s' = \psi_{n-1}(\chi_{(p,\,l)}) + p^{n-1} s$ and $t' = \psi_{n-1}(\chi_{(p,\,l)}) + p^{n-1} t$. Then $s' \equiv t' \pmod{p^{r+n-1} \mathbb{Z}_p}$ and therefore $\zeta_{p,\,l}(s') \equiv \zeta_{p,\,l}(t') \pmod{p^{r+n} \mathbb{Z}_p}$. By Definition 4.20 this gives (4.2). $\square$

Proposition 4.22 *Let $(p, l) \in \Psi_1^{\text{irr}}$ with $\Delta_{(p,l)} \neq 0$. For $n \geq 1$ we have*

$$\zeta_{p,\,l,\,n}(s) \equiv \Delta_{(p,\,l)} \, (s_{n+1} - s) \pmod{p \mathbb{Z}_p}, \quad s \in \mathbb{Z}_p$$

where s_{n+1} is defined by $\chi_{(p,\,l)} = s_2 + s_3 \, p + \ldots$. There exists the Mahler expansion

$$\zeta_{p,\,l,\,n}(s) = \zeta_{p,\,l,\,n}(0) + \sum_{\nu \geq 1} p^{n(\nu-1)} z_\nu \binom{s}{\nu}, \quad s \in \mathbb{Z}_p$$

with $z_\nu \in \mathbb{Z}_p$ where $z_\nu = (-1)^\nu p^{-n(\nu-1)} \, \mathcal{D}^\nu \zeta_{p,\,l,\,n}(0)$ and $z_1 \equiv -\Delta_{(p,\,l)} \pmod{p \mathbb{Z}_p}$.

Proof. Proposition 2.7 also works with $\alpha_j \equiv \zeta_{p,\,l,\,n}(j) \pmod{p\mathbb{Z}_p}$ for $j \in \mathbb{N}_0$. Since $\zeta_{p,\,l,\,n}(0)$ corresponds to the related irregular pair of order n, we then have

$$\zeta_{p,\,l,\,n}(0) \equiv \Delta_{(p,\,l)}\, s_{n+1} \pmod{p\mathbb{Z}_p}$$

where s_{n+1} is given by Definition 4.4. A further consequence of Proposition 2.7, extended to $s \in \mathbb{Z}_p$, is that

$$\mathcal{D}\,\zeta_{p,\,l,\,n}(s) \equiv \Delta_{(p,\,l)} \pmod{p\mathbb{Z}_p} \tag{4.3}$$

and furthermore

$$\zeta_{p,\,l,\,n}(s) \equiv \Delta_{(p,\,l)}\,(s_{n+1} - s) \pmod{p\mathbb{Z}_p}.$$

By construction $\zeta_{p,\,l,\,n}$ is continuous; this is also a consequence of (4.2). Theorem 4.19 shows that $\zeta_{p,\,l,\,n}$ has a Mahler series with the coefficients $a_\nu = (-1)^\nu\, \mathcal{D}^\nu \zeta_{p,\,l,\,n}(0)$ for $\nu \geq 0$. Using (4.1) we see that $a_\nu \in p^{n(\nu-1)}\mathbb{Z}_p$ for $\nu \geq 1$. Thus, we can set $z_\nu = p^{-n(\nu-1)} a_\nu$ for $\nu \geq 1$ to obtain the proposed series expansion above. From (4.3) we deduce that $z_1 \equiv -\Delta_{(p,\,l)} \pmod{p\mathbb{Z}_p}$. □

Corollary 4.23 *Let $(p, l) \in \Psi_1^{\mathrm{irr}}$ with $\Delta_{(p,\,l)} \neq 0$. Let*

$$\zeta_{p,\,l,\,1}(s) = \zeta_{p,\,l,\,1}(0) + \sum_{\nu \geq 1} p^{\nu-1} z_\nu \binom{s}{\nu}, \quad s \in \mathbb{Z}_p$$

be the Mahler expansion of $\zeta_{p,\,l,\,1}$ given by Proposition 4.22. We have for $r \geq 1$ that

$$\zeta_{p,\,l}(s) \equiv \zeta_{p,\,l}(0) + \sum_{\nu=1}^{r-1} p^\nu z_\nu \binom{s}{\nu} \pmod{p^r\mathbb{Z}_p}, \quad s \in \mathbb{Z}_p.$$

Special cases are given by

$$\zeta_{p,\,l}(0) = -\sum_{\nu \geq 1} p^\nu z_\nu \binom{\chi_{(p,\,l)}}{\nu}$$

and for $r \geq 1$:

$$\zeta_{p,\,l}(0) \equiv -\sum_{\nu=1}^{r-1} p^\nu z_\nu \binom{\chi_{(p,\,l)}}{\nu} \pmod{p^r\mathbb{Z}_p}.$$

Proof. We rewrite the Mahler expansion above using $\zeta_{p,\,l,\,1}(s) = p^{-1}\zeta_{p,\,l}(s)$ which yields

$$\zeta_{p,\,l}(s) = \zeta_{p,\,l}(0) + \sum_{\nu \geq 1} p^\nu z_\nu \binom{s}{\nu}, \quad s \in \mathbb{Z}_p.$$

By the assumption $\Delta_{(p,\,l)} \neq 0$ we have the zero $\chi_{(p,\,l)}$ of $\zeta_{p,\,l}$. The finite sums follow easily $\pmod{p^r\mathbb{Z}_p}$, since the coefficients $p^\nu z_\nu$ vanish for $\nu \geq r$. □

Hence, we can use Corollary 4.23 to verify calculations of the coefficients z_ν and of the zero $\chi_{(p,\,l)}$. Now, we are almost ready to prove Theorem 4.10; we first recall the following lemma from [17, p. 227].

Lemma 4.24 *For $k \geq 1$ and $p^j \leq k < p^{j+1}$, we have*

$$\left| \binom{s}{k} - \binom{t}{k} \right|_p \leq p^j \, |s-t|_p, \quad s,t \in \mathbb{Z}_p.$$

PROOF OF THEOREM 4.10. Let $s,t \in \mathbb{Z}_p$ with $s \neq t$ and set $r = \mathrm{ord}_p(s-t) \geq 0$. Then we have by Kummer congruences that

$$s \equiv t \pmod{p^r \mathbb{Z}_p} \quad \Longrightarrow \quad \zeta_{p,l}(s) \equiv \zeta_{p,l}(t) \pmod{p^{r+1}\mathbb{Z}_p}. \tag{4.4}$$

We show that

$$\frac{\zeta_{p,l}(s) - \zeta_{p,l}(t)}{p\,(s-t)} \equiv -\Delta_{(p,l)} \pmod{p\mathbb{Z}_p} \quad \text{for} \quad s \neq t \tag{4.5}$$

which also implies the converse to (4.4). Set $n = \min\{\mathrm{ord}_p\, \zeta_{p,l}(s), \mathrm{ord}_p\, \zeta_{p,l}(t)\}$ where $n \geq 1$. By Definition 4.20 we rewrite $s = \psi_{n-1}(\chi_{(p,l)}) + p^{n-1}\, s'$ and $t = \psi_{n-1}(\chi_{(p,l)}) + p^{n-1}\, t'$ with some $s', t' \in \mathbb{Z}_p$. Set $u = r+1-n \geq 0$. Note that $\mathrm{ord}_p(s'-t') = u$. By the implication in (4.4) we then have

$$p^{-n}(\zeta_{p,l}(s) - \zeta_{p,l}(t)) \equiv \zeta_{p,l,n}(s') - \zeta_{p,l,n}(t') \equiv p^u c \pmod{p^{u+1}\mathbb{Z}_p}$$

with some integer c. On the other side, we can use the Mahler expansion of $\zeta_{p,l,n}$ given by Proposition 4.22. We obtain

$$\begin{aligned} \zeta_{p,l,n}(s') - \zeta_{p,l,n}(t') \quad &\equiv \quad z_1\,(s'-t') + p^n\, z_2 \left[\binom{s'}{2} - \binom{t'}{2} \right] \\ &\qquad + p^{2n}\, z_3 \left[\binom{s'}{3} - \binom{t'}{3} \right] + \ldots \qquad \pmod{p^{u+1}\mathbb{Z}_p}. \end{aligned}$$

Since $\mathrm{ord}_p(s'-t') = u$, Lemma 4.24 shows that

$$\mathrm{ord}_p \left[\binom{s'}{k} - \binom{t'}{k} \right] \geq u - \lfloor \log_p k \rfloor$$

where $\log_p$ is the real valued logarithm with base p. Since we have $p \geq 5$ we obtain the estimate

$$\mathrm{ord}_p \left(p^{n(k-1)} z_k \left[\binom{s'}{k} - \binom{t'}{k} \right] \right) \geq u + n(k-1) - \lfloor \log_p k \rfloor \geq u+1$$

for all $k \geq 2$. Hence, all terms vanish $\pmod{p^{u+1}}$ except for $k=1$:

$$\zeta_{p,l,n}(s') - \zeta_{p,l,n}(t') \equiv z_1\,(s'-t') \pmod{p^{u+1}\mathbb{Z}_p}.$$

By Proposition 4.22 and $\mathrm{ord}_p(s'-t') = u$ we finally get

$$z_1\,(s'-t') \equiv -\Delta_{(p,l)}(s'-t') \equiv p^u c \pmod{p^{u+1}\mathbb{Z}_p}$$

which shows that $c \not\equiv 0 \pmod{p}$. Since $p^{-n+1}\,(s-t) = s'-t'$, this deduces (4.5).

Now, we have to determine the derivative of $\zeta_{p,l}$. Taking any sequence $(t_\nu)_{\nu \geq 1}$ with $t_\nu \neq s$ for all $\nu \geq 1$ and $\lim_{\nu \to \infty} t_\nu = s$, (4.5) yields the derivative with

$$\zeta'_{p,l}(s) \equiv -p\,\Delta_{(p,l)} \pmod{p^2\mathbb{Z}_p}.$$

□

Remark 4.25 In general the converse of the Kummer congruences does not hold. The first nontrivial counterexample is given by $p = 13$ and $B_{16}/16 - B_4/4 = -7 \cdot 13^2/2720$. The only case where the divided Bernoulli numbers are equal is $B_{14}/14 - B_2/2 = 0$.

At the end of this section we revisit the p-adic zeta functions $\zeta_{p,\,s_1}$ in the case $s_1 \neq 0$ as introduced by Definition 4.2. We can transfer some results to these functions. Moreover, we can state a formula equivalent to the Kummer congruences, but which involves values of p-adic zeta functions at smallest possible argument values.

Proposition 4.26 *Let p be a prime with $p \geq 5$ and $s_1 \in \{2, 4, \ldots, p-3\}$ be fixed. Then*

$$\mathcal{D}^r \zeta_{p,\,s_1}(s) \equiv 0 \pmod{p^r \mathbb{Z}_p}, \quad s \in \mathbb{Z}_p. \tag{4.6}$$

There exists the Mahler expansion

$$\zeta_{p,\,s_1}(s) = \zeta_{p,\,s_1}(0) + \sum_{\nu \geq 1} p^\nu z_\nu \binom{s}{\nu}, \quad s \in \mathbb{Z}_p \tag{4.7}$$

with $z_\nu \in \mathbb{Z}_p$ where $z_\nu = (-1)^\nu p^{-\nu}\, \mathcal{D}^\nu \zeta_{p,\,s_1}(0)$.

Proof. In analogy to Propositions 4.21 and 4.22 in the case $n = 1$, Congruence (4.6) is a consequence of Theorem 2.4. Since $\zeta_{p,\,s_1}$ is a continuous function on $\mathbb{Z}_p$, we obtain a Mahler expansion where the proposed coefficients follow by (4.6). $\square$

Theorem 4.27 *Let p be a prime with $p \geq 5$ and $s_1 \in \{2, 4, \ldots, p-3\}$ be fixed. For $r \geq 0$ we have*

$$\zeta_{p,\,s_1}(s) \equiv \sum_{k=0}^{r} \zeta_{p,\,s_1}(k)\; T_{r,k}(s) \pmod{p^{r+1} \mathbb{Z}_p}, \quad s \in \mathbb{Z}_p$$

where the polynomials $T_{r,k} : \mathbb{Z}_p \to \mathbb{Z}_p$ with $T_{r,k} \in \mathbb{Q}[x]$ and $\deg T_{r,k} = r$ are given by

$$T_{r,k}(x) = \sum_{j=k}^{r} (-1)^{j+k} \binom{j}{k} \binom{x}{j}.$$

Proof. We rewrite the Mahler expansion of $\zeta_{p,\,s_1}$ given in (4.7). Let $s \in \mathbb{Z}_p$. For $r \geq 0$ we obtain the finite expansion

$$\zeta_{p,\,s_1}(s) \equiv \zeta_{p,\,s_1}(0) + \sum_{\nu=1}^{r} p^\nu z_\nu \binom{s}{\nu} \equiv \sum_{j=0}^{r} (-1)^j\, \mathcal{D}^j \zeta_{p,\,s_1}(0) \binom{s}{j} \pmod{p^{r+1} \mathbb{Z}_p}.$$

By Definition 4.18 we have

$$\mathcal{D}^j \zeta_{p,\,s_1}(0) = \sum_{k=0}^{j} \binom{j}{k} (-1)^k \zeta_{p,\,s_1}(k) = \sum_{k=0}^{r} \binom{j}{k} (-1)^k \zeta_{p,\,s_1}(k).$$

Reordering the finite sums and terms yields the definition of $T_{r,k}$ above. Since $x \mapsto \binom{x}{j}$ which is a polynomial of degree j defines a function on $\mathbb{Z}_p$, the polynomials $T_{r,k}$ are functions on $\mathbb{Z}_p$. The coefficients of $T_{r,k}$ are rational, e.g., rewrite $T_{r,r}(x) = \binom{x}{r}$ as a polynomial in x. We deduce that $\deg T_{r,k} = r$, because the term $\binom{x}{r}$ which occurs only once gives the maximal degree. $\square$

Corollary 4.28 *Let n be an even positive integer and p be a prime where $p-1 \nmid n$. Define the integer s_1 by $s_1 \equiv n \pmod{p-1}$ with $0 < s_1 < p-1$. Set $s = (n-s_1)/(p-1)$. For $r \geq 1$ we have*

$$(1-p^{n-1})\widehat{B}(n) \equiv \sum_{k=0}^{r-1}(1-p^{s_1+k(p-1)-1})\widehat{B}(s_1+k(p-1))\ T_{r-1,k}(s) \pmod{p^r\mathbb{Z}_p}$$

with $T_{r,k}$ as defined above.

Proof. This is a reformulation of Theorem 4.27 using Definition 4.2 where $s_1 + s(p-1) = n$. $\square$

Remark 4.29 The case $r = 1$ of Corollary 4.28 reduces to a special case of the Kummer congruences (1.3) for $r = 1$. We explicitly give the cases $r = 2, 3, 4$ of Corollary 4.28. Note that $s_1 \geq 2$ and $p \geq 5$. Case $r = 2$:

$$(1-p^{n-1})\widehat{B}(n) \equiv -(s-1)\,(1-p^{s_1-1})\widehat{B}(s_1) + s\,\widehat{B}(s_1+p-1) \pmod{p^2\mathbb{Z}_p}.$$

Case $r = 3$:

$$\begin{aligned}(1-p^{n-1})\widehat{B}(n) \equiv\ & +\frac{1}{2}(s^2-3s+2)\,(1-p^{s_1-1})\widehat{B}(s_1)\\ & -s(s-2)\,\widehat{B}(s_1+p-1)\\ & +\frac{1}{2}s(s-1)\,\widehat{B}(s_1+2(p-1)) \pmod{p^3\mathbb{Z}_p}.\end{aligned}$$

Case $r = 4$:

$$\begin{aligned}(1-p^{n-1})\widehat{B}(n) \equiv\ & -\frac{1}{6}(s^3-6s^2+11s-6)\,(1-p^{s_1-1})\widehat{B}(s_1)\\ & +\frac{1}{2}s(s^2-5s+6)\,\widehat{B}(s_1+p-1)\\ & -\frac{1}{2}s(s^2-4s+3)\,\widehat{B}(s_1+2(p-1))\\ & +\frac{1}{6}s(s-1)(s-2)\,\widehat{B}(s_1+3(p-1)) \pmod{p^4\mathbb{Z}_p}.\end{aligned}$$

5 Algorithms

Here we will give some algorithms for calculating irregular pairs of higher order assuming we are in the nonsingular case $\Delta_{(p,l)} \neq 0$. As a result of Theorem 3.1 and Theorem 4.1, one has first to calculate $\Delta_{(p,l)}$, then the irregular pair of order n resp. the corresponding divided Bernoulli number provides the next related irregular pair of order $n+1$. This is not practicable for higher orders say $n > 3$. Proposition 2.10 shows a more effective way to determine related irregular pairs of higher order. Starting from an irregular pair $(p, l) \in \Psi_n^{\mathrm{irr}}$ with $n \geq 1$ and requiring that $l > (r-1)n$ with some $r \geq 2$ we can obtain a related irregular pair $(p, l') \in \Psi_{rn}^{\mathrm{irr}}$. If the corresponding sequence $(\alpha_j)_{j \geq 0}$ is equidistant $(\mathrm{mod}\ p^{(r-1)n})$, then one can easily apply this proposition. If not, one has to calculate successively all elements α_j in order to find $\alpha_s \equiv 0 \pmod{p^{(r-1)n}}$ where $0 \leq s < p^{(r-1)n}$ which exists uniquely by the assumption $\Delta_{(p,l)} \neq 0$. To shorten these calculations, this search can be accomplished step by step, moving each time from a sequence $(\alpha_{j,k})_{j \geq 0}$ to a sequence $(\alpha_{j,k+1})_{j \geq 0}$ which are assigned to the irregular pair of order k resp. $k+1$.

Proposition 5.1 *Let $(p, l) \in \Psi_n^{\mathrm{irr}}$, $n \geq 1$ with $\Delta_{(p,l)} \neq 0$. Let r, u be positive integers with $r > 1$ and $u = (r-1)n$. Assume that $l > u$. Let the elements*

$$\alpha_{j,0} \equiv p^{-n} \widehat{B}(l + j\varphi(p^n)) \pmod{p^u}, \quad j = 0, \ldots, r-1$$

be given. For each step $k = 0, \ldots, u-1$ proceed as follows:
The elements $\alpha_{j,k}$ with $j = 0, \ldots, rp-1$ have to be calculated successively by

$$\alpha_{j+r,k} \equiv (-1)^{r+1} \sum_{\nu=0}^{r-1} \binom{r}{\nu} (-1)^\nu \alpha_{j+\nu,k} \pmod{p^{u-k}}.$$

Set $s_k \equiv -\alpha_{0,k} \Delta_{(p,l)}^{-1} \pmod{p}$ with $0 \leq s_k < p$. The elements $\alpha_{j,k}$ which are divisible by p are given by

$$\alpha_{s_k + \mu p, k} \equiv 0 \pmod{p}, \quad \mu = 0, \ldots, r-1.$$

For $k < u-1$ set

$$\alpha_{j,k+1} = \alpha_{s_k + jp, k}/p, \quad j = 0, \ldots, r-1$$

and go to the next step $k+1$, otherwise stop. Let $(p, t_1, \ldots, t_n) \in \widehat{\Psi}_n^{\mathrm{irr}}$ be the associated irregular pair with (p, l), then

$$(p, t_1, \ldots, t_n, s_0, \ldots, s_{u-1}) \in \widehat{\Psi}_{rn}^{\mathrm{irr}}$$

is the only related irregular pair of order rn.

Proof. Proposition 2.10 shows that

$$\sum_{\nu=0}^{r} \binom{r}{\nu} (-1)^\nu \alpha_{j+\nu,0} \equiv 0 \pmod{p^u}. \tag{5.1}$$

All elements of the sequence $(\alpha_{j,0})_{j\geq 0}$ can be calculated successively induced by the first elements $\alpha_{j,0}$ with $j = 0, \ldots, r-1$. Using Corollary 2.5 with $\omega = p^k \varphi(p^n)$, $0 \leq k < u$, then (5.1) becomes

$$\sum_{\nu=0}^{r} \binom{r}{\nu} (-1)^\nu \alpha_{j+\nu p^k,0} \equiv 0 \pmod{p^u}, \tag{5.2}$$

whereby the sequence $(\alpha_{j+\mu p^k,0})_{\mu\geq 0}$ can also be calculated successively. Note that now j is fixed and μ runs. The sequences $(\alpha_{j,k})_{j\geq 0}$ which we will consider are subsequences of $(\alpha_{j,0})_{j\geq 0}$ in a suitable manner. Essentially, these sequences are given by (5.2). The existence of these sequences and that they correspond to the related irregular pair of order $n+k$ will be shown by induction on k for $k = 0, \ldots, u-1$. Set $l_n = l$. By Proposition 2.7 and Theorem 3.1 there exist certain integers s_k and related irregular pairs of higher order for $k = 0, \ldots, u-1$ where

$$(p, l_{n+k+1}) \in \Psi^{\mathrm{irr}}_{n+k+1}, \quad l_{n+k+1} = l_{n+k} + s_k \, \varphi(p^{n+k}) \quad \text{with} \quad 0 \leq s_k < p.$$

Basis of induction $k = 0$: The sequence $(\alpha_{j,0})_{j\geq 0}$ is given by (5.1) and we have

$$\alpha_{j,0} \equiv p^{-n} \, \widehat{B}(l_n + j\varphi(p^n)) \pmod{p^u}.$$

Inductive step $k \mapsto k+1$: Assume true for k prove for $k+1$. The elements $\alpha_{j,k}$ with $j = 0, \ldots, r-1$ are given and the following elements are calculated by

$$\alpha_{j+r,k} \equiv (-1)^{r+1} \sum_{\nu=0}^{r-1} \binom{r}{\nu} (-1)^\nu \alpha_{j+\nu,k} \pmod{p^{u-k}} \tag{5.3}$$

up to index $j = rp - 1$. Proposition 2.7 provides

$$s_k \equiv -\alpha_{0,k} \Delta^{-1}_{(p,\,l)} \pmod{p} \quad \text{with} \quad 0 \leq s_k < p. \tag{5.4}$$

In the case $k < u-1$, Lemma 2.6 additionally ensures that only $\alpha_{s_k+jp,k} \equiv 0 \pmod{p}$ for all j. Thus, we can define a new sequence by

$$\begin{aligned} \alpha_{j,k+1} &\equiv \alpha_{s_k+jp,k}/p \\ &\equiv p^{-(n+k+1)} \, \widehat{B}(l_{n+k} + (s_k + jp)\varphi(p^{n+k})) \\ &\equiv p^{-(n+k+1)} \, \widehat{B}(l_{n+k+1} + j\varphi(p^{n+k+1})) \qquad \pmod{p^{u-(k+1)}} \end{aligned} \tag{5.5}$$

for $j = 0, \ldots, r-1$. By definition $(p\,\alpha_{j,k+1})_{j=0,\ldots,r-1}$ is a subsequence of $(\alpha_{j,k})_{j\geq 0}$. Inductively $(p^{k+1}\,\alpha_{j,k+1})_{j=0,\ldots,r-1}$ is a subsequence of $(\alpha_{j,0})_{j\geq 0}$ and satisfies (5.2) in a suitable manner and therefore also (5.3) considering case $k+1$. On the other side, Congruence (5.5) shows that the new sequence also corresponds to the related irregular pair of order $n+k+1$.

Let $(p, t_1, \ldots, t_n) \in \widehat{\Psi}^{\mathrm{irr}}_n$ be the associated irregular pair with (p, l). Congruence (5.4) provides a unique integer s_k for each step. Thus, $(p, t_1, \ldots, t_n, s_0, \ldots, s_{u-1}) \in \widehat{\Psi}^{\mathrm{irr}}_{rn}$ is the only related irregular pair of order rn. $\square$

Remark 5.2 Unfortunately, the previous proposition has the restriction that for an irregular pair $(p, l) \in \Psi_n^{\mathrm{irr}}$ of order n and parameter r the following must hold that

$$l > (r-1)n.$$

Consider $(691, 12) \in \Psi_1^{\mathrm{irr}}$. In this case one only could calculate related irregular pairs up to order 12. However, this restriction can be removed by shifting the index of the starting sequence $(\alpha_{j,0})_{j \geq 0}$. Then shifting $j \mapsto j + t$ yields

$$l + t\,\varphi(p^n) > (r-1)n$$

enabling one to choose a greater value of r. In general, one has to proceed in the following way. Moving from a sequence $(\alpha_{j,k})_{j \geq 0}$ to $(\alpha_{j,k+1})_{j \geq 0}$ one has to determine elements $\alpha_{j,k} \equiv 0 \pmod{p}$. If one starts with a shifted sequence $(\alpha'_{j,k})_{j \geq 0} = (\alpha_{j,k})_{j \geq t}$ having elements $\alpha_{j,k} \equiv 0 \pmod{p}$ with $0 \leq j < t$, then the calculated sequence $(\alpha'_{j,k+1})_{j \geq 0}$ is also shifted in the index compared to $(\alpha_{j,k+1})_{j \geq 0}$. This can be easily corrected by comparing the sequences resp. the resulting integers s_k with unshifted sequences calculated with a lower $r' < r$. In this case determining the integer s_k is better done by searching $\alpha_{j,k} \equiv 0 \pmod{p}$ rather than calculating via (5.4).

The main result can be stated as follows: irregular pairs of higher order can be determined with little effort by calculating a small number of divided Bernoulli numbers with small indices. We shall give another algorithm in terms of the p-adic zeta function $\zeta_{p,l}$ which produces a truncated p-adic expansion of $\chi_{(p,l)}$.

Proposition 5.3 *Let $(p, l) \in \Psi_1^{\mathrm{irr}}$ with $\Delta_{(p,l)} \neq 0$. Let n be a positive integer. Initially calculate the values*

$$\zeta_{p,l,1}(k) \equiv p^{-1}\zeta_{p,l}(k) \pmod{p^n\mathbb{Z}_p}$$

for $k = 0, \ldots, n$ and

$$\Delta_{(p,l)} \equiv \zeta_{p,l,1}(0) - \zeta_{p,l,1}(1) \pmod{p\mathbb{Z}_p}$$

where $0 < \Delta_{(p,l)} < p$. Set $t_1 = 0$. For each step $r = 1, \ldots, n$ proceed as follows: Calculate

$$\xi_r \equiv \sum_{k=0}^{r} \zeta_{p,l,1}(k)\, T_{r,k}(t_r) \pmod{p^r\mathbb{Z}_p}$$

with the polynomials $T_{r,k}$ as defined in Theorem 4.27. Then $\xi_r \in p^{r-1}\mathbb{Z}_p$. Set

$$s_{r+1} \equiv \Delta_{(p,l)}^{-1}\, p^{1-r}\, \xi_r \pmod{p\mathbb{Z}_p}$$

where $0 \leq s_{r+1} < p$. Set $t_{r+1} = t_r + s_{r+1}\, p^{r-1}$ and go to the next step while $r < n$. Finally $t_{n+1} \equiv \chi_{(p,l)} \pmod{p^n\mathbb{Z}_p}$.

Proof. By Definition 4.20 we have $\zeta_{p,l,1}(k) = p^{-1}\zeta_{p,l}(k)$. The value $\Delta_{(p,l)}$ is given by its definition. Define $t'_r = \psi_{r-1}(\chi_{(p,l)})$ for any $r \geq 1$ where the expansion of the zero of $\zeta_{p,l}$ is given by

$$\chi_{(p,l)} = s_2 + s_3\, p + \ldots + s_{r+1}\, p^{r-1} + \ldots.$$

For now, let $r \in \{1, \ldots, n\}$ be fixed. Theorem 4.27 provides

$$\zeta_{p,\,l,\,1}(s) \equiv \sum_{k=0}^{r} \zeta_{p,\,l,\,1}(k)\; T_{r,k}(s) \pmod{p^r \mathbb{Z}_p}$$

for $s \in \mathbb{Z}_p$. From Definition 4.20 we have

$$\zeta_{p,\,l,\,r}(0) = p^{-r} \zeta_{p,\,l}(\psi_{r-1}(\chi_{(p,\,l)})) = p^{1-r} \zeta_{p,\,l,\,1}(t'_r).$$

Proposition 4.22 shows that

$$\zeta_{p,\,l,\,r}(0) \equiv \Delta_{(p,\,l)}\, s_{r+1} \pmod{p\mathbb{Z}_p}.$$

By construction we have $t'_{r+1} = t'_r + s_{r+1}\, p^{r-1}$. Since $t'_1 = t_1 = 0$ we deduce by induction on r that $t'_r = t_r$ and $\xi_r \equiv \zeta_{p,\,l,\,1}(t'_r) \pmod{p^r \mathbb{Z}_p}$ for $1 \leq r \leq n$. This produces $t_{n+1} = \psi_n(\chi_{(p,\,l)})$ which equals to $t_{n+1} \equiv \chi_{(p,\,l)} \pmod{p^n \mathbb{Z}_p}$. Since $1 \leq r \leq n$, we need the values $\zeta_{p,\,l,\,1}(k) \pmod{p^n \mathbb{Z}_p}$ for $k = 0, \ldots, n$. $\square$

Proposition 5.4 *Let n be an even positive integer, then*

$$B_n = (-1)^{\frac{n}{2}-1} \prod_{p-1 \nmid n} p^{\tau(p,n)+\operatorname{ord}_p n} \Big/ \prod_{p-1 \mid n} p$$

where

$$\tau(p,n) := \sum_{\nu=1}^{\infty} \#(\, \Psi_\nu^{\text{irr}} \cap \{(p, n \bmod \varphi(p^\nu))\}\,).$$

Here, as in Definition 2.1, $x \bmod y$ denotes the least nonnegative residue of x modulo y.

Proof. The trivial parts of the products above are given by (1.4), (1.5), and the sign. Thus, the product $\prod_{p-1 \nmid n} p^{\tau(p,n)}$ remains. From Definition 2.1 and Remark 2.2, the function $\tau(p,n)$ follows by applying the maps λ_ν resp. Kummer congruences which results in a chain of related irregular pairs of descending order similar to (2.1). $\square$

The previous proposition gives an unconditional representation of the Bernoulli numbers by means of the sets Ψ_ν^{irr}. Theorem 4.8 also gives a representation by zeros $\chi_{(p,\,l)}$ assuming the Δ-Conjecture. Of course, the problem of determining the occurring irregular prime factors remains open. On the other side, for instance, if one has calculated the first irregular pairs of order 10 for the first irregular primes $p_1, \ldots, p_r$ like Table A.3, then one can specify *ad hoc* all irregular prime powers $p_\nu^{e_\nu}$ with $p_\nu \leq p_r$ of B_n resp. $\zeta(1-n)$ up to index $n = 4 \cdot 10^{15}$. Note that this lower bound is here determined by the first irregular prime 37 and order 10.

Define for positive integers n and m the summation function of consecutive integer powers by $S_n(m) = \sum_{\nu=0}^{m-1} \nu^n$. Many congruences concerning the function S_n are naturally related to the Bernoulli numbers.

Proposition 5.5 *Let $(p,l) \in \Psi_1^{\mathrm{irr}}$. Assume that $n \equiv l \pmod{p-1}$ where $n > 0$, then*

$$\frac{B_n}{n} \equiv \frac{S_n(p)}{n\,p} \pmod{p^2}. \tag{5.6}$$

Moreover

$$\Delta_{(p,\,l)} \equiv p^{-2} \left(\frac{S_{l+p-1}(p)}{l-1} - \frac{S_l(p)}{l} \right) \pmod{p} \tag{5.7}$$

with $0 \leq \Delta_{(p,\,l)} < p$.

Proof. Let $n \equiv l \pmod{p-1}$. The well-known formula of S_n, see [9, p. 234], is given by

$$\frac{S_n(p)}{n\,p} = \frac{B_n}{n} + \binom{n-1}{1} B_{n-2} \frac{p^2}{2 \cdot 3} + \sum_{k=3}^{n} \binom{n-1}{k-1} B_{n-k} \frac{p^k}{k(k+1)} \tag{5.8}$$

where the equation is divided by n and p. Note that $p \geq 37$ and $n \geq l \geq 12$, because 37 is the first irregular prime and $B_{12}/12$ is the first divided Bernoulli number which has a numerator greater one. Now, the properties of (1.4) and (1.5) provide that B_n/n and B_{n-2} are p–integral. For the other terms with $B_{n-k} \neq 0$ it follows that pB_{n-k} is p–integral and $\mathrm{ord}_p\,(p^{k-1}/(k(k+1))) \geq 2$ by a standard counting argument. Therefore, Equation (5.8) is p-integral and holds $\pmod{p^2}$ whereas all terms of the right side vanish except for B_n/n. This gives Congruence (5.6). From Definition 2.3 we have

$$p\,\Delta_{(p,\,l)} \;\equiv\; \widehat{B}(l+p-1) - \widehat{B}(l) \pmod{p^2}.$$

We can apply (5.6) in the congruence above. Reducing a p-power and considering that $l+p-1 \equiv l-1 \not\equiv 0 \pmod{p}$ finally yields (5.7). □

Looking at each line of Table A.3, the product of the first three entries $\Delta_{(p,\,l)}$, s_1, and s_2 are connected with the function S_n. Thus, one can easily verify these values.

Proposition 5.6 *Let $(p,l) \in \Psi_1^{\mathrm{irr}}$ with $\Delta_{(p,\,l)} \neq 0$. Let $(p, s_1, s_2) \in \widehat{\Psi}_2^{\mathrm{irr}}$ be the related irregular pair of order two with $l = s_1$. Then*

$$\Delta_{(p,\,l)}\, s_1\, s_2 \equiv -p^{-2}\, S_l(p) \pmod{p}.$$

Proof. By Proposition 2.7 we have

$$s_2 \equiv -p^{-1} \frac{B_l}{l} \Delta_{(p,\,l)}^{-1} \pmod{p}.$$

We then obtain by Proposition 5.5 that

$$p\,s_2\,\Delta_{(p,\,l)} \equiv -\frac{B_l}{l} \equiv -\frac{S_l(p)}{l\,p} \pmod{p^2}$$

which deduces the result since $s_1 = l < p$. □

Now, we shall give some reasons why a prediction or description of the occurrence of irregular prime factors of Bernoulli numbers seems to be impossible in general. For example, we have with an extremely small index $n = 42$ that

$$B_{42} = \frac{1520097643918070802691}{1806}$$

observing that the numerator is a large irregular prime! As mentioned in Section 3, Bernoulli numbers B_n with index up to $n = 152$ have large irregular prime factors with 30 up to 100 digits. This is even now the greatest mystery of the Bernoulli numbers!

The connection with the Riemann zeta function $\zeta(s)$ via (1.1) leads to methods of calculating Bernoulli numbers directly in a fast and effective way, see [11, Section 2.7], noting that the main part of the calculation can be done using integers only. Let n be an even positive integer and $|B_n| = U_n/V_n$ with $(U_n, V_n) = 1$ then (1.1) reads

$$U_n = \tau_n \, \zeta(n), \qquad \tau_n = 2V_n \frac{n!}{(2\pi)^n}, \qquad V_n = \prod_{p-1|n} p$$

with V_n given by (1.4). Since $\zeta(n) \to 1$ for $n \to \infty$, τ_n is a first approximation of the numerator U_n. Considering the decimal digit representation of U_n and τ_n, about $n/3$ digits of the most significant decimal digits of U_n and τ_n are equal, see [11, Satz 2.7.9, p. 75] for a more precise statement and formula. Visiting B_{42} again, we observe 12 identical digits:

$$\begin{aligned} U_{42} &= \mathbf{1\,520\,097\,643\,91}8\,070\,802\,691, \\ \tau_{42} = 2 \cdot 1806 \cdot 42!/(2\pi)^{42} &\approx \mathbf{1\,520\,097\,643\,91}7\,725\,172\,488.7773\,. \end{aligned}$$

How can we interpret this result? A part of the most significant digits of the numerator of the Bernoulli number B_n is determined in a certain way by all primes $p \leq n+1$ and the reciprocal n-th power of π. That the digits of π are involved in the numerators of the Bernoulli numbers is quite remarkable.

6 Connections with Iwasawa theory

The Δ-Conjecture is directly connected with Iwasawa theory of cyclotomic fields over $\mathbb{Q}$. Let p be an odd prime and μ_{p^n} be the set of p^n-th roots of unity where n is a positive integer. For the cyclotomic field $\mathbb{Q}(\mu_{p^n})$ let $\mathbb{Q}(\mu_{p^n})^+$ be its maximal real subfield. The class number $h_p = h(\mathbb{Q}(\mu_p))$ can be factored by $h_p = h_p^- \, h_p^+$ where $h_p^+ = h(\mathbb{Q}(\mu_p)^+)$ and h_p^- is the relative class number introduced by Kummer. Define

$$B_{1,\omega^m} = \frac{1}{p} \sum_{a=1}^{p-1} a \, \omega^m(a)$$

as the generalized Bernoulli number assigned to the Teichmüller character ω. This character is defined by $\omega : \mathbb{Z}_p^* \to \mathbb{Z}_p^*$ and $\omega(a) \equiv a \pmod{p\mathbb{Z}_p}$ for $a \in \mathbb{Z}_p^*$ giving the

$(p-1)$-th roots of unity in $\mathbb{Q}_p$. We have for even $m > 0$ and $p-1 \nmid m$ the following relation, see [23, Corollary 5.15, p. 61]:

$$B_{1,\omega^{m-1}} \equiv \widehat{B}(m) \pmod{p\mathbb{Z}_p}.$$

For the detailed theory, especially of Iwasawa invariants and cyclotomic $\mathbb{Z}_p$-extensions, see Washington [23] and Greenberg [8]. The results of Iwasawa, Ferrero and Washington, Vandiver and Kummer provide the following theorem, see [23, Cor. 10.17, p. 202].

Theorem 6.1 *Let p be an irregular prime. Assume the following conditions for all irregular pairs (p,l):*

(1) The conjecture of Kummer–Vandiver holds: $p \nmid h_p^+$,

(2) The Kummer congruence does not hold $\pmod{p^2}$: $\widehat{B}(l+p-1) \not\equiv \widehat{B}(l) \pmod{p^2}$,

(3) The generalized Bernoulli number is not divisible by p^2: $B_{1,\omega^{l-1}} \not\equiv 0 \pmod{p^2\mathbb{Z}_p}$.

If these are satisfied, then

$$\operatorname{ord}_p h(\mathbb{Q}(\mu_{p^n})) \;=\; i(p)\, n \quad \textit{for all} \quad n \geq 1.$$

All conditions of the theorem above hold for all irregular primes $p < 12\,000\,000$ as verified in [2]. In the case of a regular prime p the formula of the theorem above is also valid, because then we have $i(p) = 0$ and

$$p \nmid h_p = h(\mathbb{Q}(\mu_p)) \quad \Longleftrightarrow \quad p \nmid h(\mathbb{Q}(\mu_{p^n})) \quad \text{for all} \quad n \geq 1$$

as follows, e.g., from Iwasawa theory. To get another point of view we can exchange two conditions of the previous theorem by our results. Conditions equivalent to those of Theorem 6.1 are as follows:

(2') The Δ-Conjecture holds: $\Delta_{(p,\,l)} \neq 0$,

(3') A special irregular pair of order two does not exist: $(p,l,l-1) \notin \widehat{\Psi}_2^{\mathrm{irr}}$.

Now, the Δ-Conjecture with its consequences gives a significant reason to believe that the condition (2) resp. (2') may hold in general. We still have to show that the condition (3') is equivalent to the condition (3).

Proposition 6.2 *Let $(p,l) \in \Psi_1^{\mathrm{irr}}$. Then*

$$B_{1,\omega^{l-1}} \equiv B_{l+(p-1)(l-1)} \pmod{p^2\mathbb{Z}_p}$$

and

$$B_{1,\omega^{l-1}} \equiv 0 \pmod{p^2\mathbb{Z}_p} \quad \Longleftrightarrow \quad (p,l,l-1) \in \widehat{\Psi}_2^{\mathrm{irr}}.$$

To prove this result, we first need some properties of the Teichmüller character ω. Since $\omega(a)$ is defined by $\omega(a) = \lim_{n\to\infty} a^{p^n}$ in $\mathbb{Z}_p$, the following lemma is easily derived.

Lemma 6.3 *Let a, p be integers with p an odd prime and $0 < a < p$. Then*

$$\omega(a) \equiv a^p \pmod{p^2\mathbb{Z}_p} \qquad \textit{and} \qquad \omega(a) \equiv a^p + p\,(a^p - a) \pmod{p^3\mathbb{Z}_p}.$$

Proof of Proposition 6.2. Since $\omega^{l-1}(a) = \omega(a^{l-1})$, we have

$$p\,B_{1,\omega^{l-1}} = \sum_{a=1}^{p-1} a\,\omega^{l-1}(a) = \sum_{a=1}^{p-1} a\,\omega(a^{l-1}).$$

Using Lemma 6.3 we obtain

$$a\,\omega(a^{l-1}) \equiv a^{p(l-1)+1} + p\,(a^{p(l-1)+1} - a^l) \pmod{p^3\mathbb{Z}_p}.$$

From the definition of S_n and Proposition 5.5 we have

$$pB_{1,\omega^{l-1}} \equiv S_{p(l-1)+1}(p) + pS_{p(l-1)+1}(p) - pS_l(p) \equiv pB_{l+(p-1)(l-1)} \pmod{p^3\mathbb{Z}_p}.$$

Since $p(l-1)+1 = l + (p-1)(l-1) \equiv l \pmod{p-1}$ and $(p,l) \in \Psi_1^{\mathrm{irr}}$, only the first term $S_{p(l-1)+1}(p)$ does not vanish. Note that $p \nmid p(l-1)+1$, therefore we get

$$0 \equiv B_{l+(p-1)(l-1)} \equiv \widehat{B}(l + (p-1)(l-1)) \pmod{p^2}$$

if and only if $(p,l,l-1) \in \widehat{\Psi}_2^{\mathrm{irr}}$. $\square$

Remark 6.4 From Propositions 5.5 and 5.6, the conditions $\Delta_{(p,l)} \neq 0$ and $(p,l,l-1) \notin \widehat{\Psi}_2^{\mathrm{irr}}$ are equivalent to the system

$$\begin{aligned} l\,S_{l+p-1}(p) - (l-1)\,S_l(p) &\not\equiv 0 \pmod{p^3}, \\ l\,S_{l+p-1}(p) - (l-2)\,S_l(p) &\not\equiv 0 \pmod{p^3}. \end{aligned}$$

7 The singular case

In Section 4 we have derived most of the results assuming the Δ-Conjecture. Theorem 4.8 conjecturally describes a closed formula for $\zeta(1-n)$ by zeros $\chi_{(p,l)}$. The following theorem gives an equivalent formulation for the Bernoulli numbers.

Theorem 7.1 *Let n be an even positive integer. Under the assumption of the Δ-Conjecture we have*

$$B_n = (-1)^{\frac{n}{2}-1} \prod_{p-1 \nmid n} |n|_p^{-1} \prod_{\substack{(p,l)\in\Psi_1^{\mathrm{irr}} \\ l \equiv n \ (\mathrm{mod}\ p-1)}} |p\,(\chi_{(p,l)} - \tfrac{n-l}{p-1})|_p^{-1} \prod_{p-1 \mid n} p^{-1}.$$

PROOF. We have to modify the formula of Theorem 4.8. The product formula gives

$$1 = \prod_{p \in \mathbb{P} \cup \{\infty\}} |n|_p = |n|_\infty \prod_{p-1 \mid n} |n|_p \prod_{p-1 \nmid n} |n|_p \,. \tag{7.1}$$

Since $-B_n/n = \zeta(1-n)$, the proposed formula follows easily. $\square$

To get an unconditional formula for B_n resp. $\zeta(1-n)$ we have to include the case of a singular $\Delta_{(p,\,l)}$. However, no such singular $\Delta_{(p,\,l)}$ has been found yet. Theorem 3.2 describes the more complicated behavior of related irregular pairs of higher order in the singular case which can be described by a rooted p-ary tree, see Diagram 3.3.

Let $(p,l) \in \Psi_1^{\mathrm{irr}}$ with a singular $\Delta_{(p,\,l)}$. We construct the rooted p-ary tree of related irregular pairs of higher order which is a consequence of Theorem 3.2. Each node contains one related irregular pair of higher order. Note that these pairs are not necessarily distinct. We denote this tree by $T^0_{(p,\,l)}$ assigned to the root node (p,l).

The tree $T^0_{(p,\,l)}$ has the property that each node of height r lies in $\Psi_{r+1}^{\mathrm{irr}}$. A tree $T^0_{(p,\,l)} = \{(p,l)\}$ is called a *trivial tree* having height 0. If the tree $T^0_{(p,\,l)}$ is of height ≥ 1, then it contains the root node (p,l) and its p child nodes $(p, l + j\varphi(p)) \in \Psi_2^{\mathrm{irr}}$ for $j = 0, \ldots, p-1$.

In the nonsingular case, we have a zero of the p-adic zeta function. In contrast the singular case does not guarantee that related irregular pairs of higher order exist at all. The discovery of a singular $\Delta_{(p,\,l)}$ is not incompatible with Theorem 7.1 but the formula becomes more complicated, because we then have to consider the complete tree $T^0_{(p,\,l)}$. By combining both cases we obtain an unconditional formula which is given by the following theorem. Recall Definition 2.1.

Theorem 7.2 *Let n be an even positive integer, then*

$$\begin{aligned} B_n \;&=\; (-1)^{\frac{n}{2}-1} \prod_{p-1 \nmid n} |n|_p^{-1} \prod_{\substack{(p,l)\in\Psi_1^{\mathrm{irr}},\, \Delta_{(p,\,l)}\neq 0 \\ l \equiv n \;(\mathrm{mod}\;\; p-1)}} |p\,(\chi_{(p,\,l)} - \tfrac{n-l}{p-1})|_p^{-1} \\ &\qquad \prod_{\substack{(p,l)\in\Psi_1^{\mathrm{irr}},\, \Delta_{(p,\,l)}= 0 \\ l \equiv n \;(\mathrm{mod}\;\; p-1)}} p^{1+h^0_{(p,\,l)}(n)} \prod_{p-1 \mid n} p^{-1} \end{aligned}$$

with the height $h^0_{(p,\,l)}$ of n defined by

$$h^0_{(p,\,l)}(n) = \max\left\{ \mathrm{height}((p,l')) : \, (p,l') \in T^0_{(p,\,l)} \cap \{(p, n \bmod \varphi(p^\nu))\}_{\nu \geq 1} \right\}.$$

Moreover, $h^0_{(p,\,l)}(n) = 0 \iff$ the tree $T^0_{(p,\,l)}$ is trivial.

PROOF. The case $\Delta_{(p,\,l)} \neq 0$ is already covered by Theorem 7.1. Next we assume that $\Delta_{(p,\,l)} = 0$ with a given tree $T^0_{(p,\,l)}$ where $n \equiv l \pmod{p-1}$. As a consequence of the construction of $T^0_{(p,\,l)}$ and Remark 2.2, we have to determine the maximal height of a node $(p, l_{\nu,j}) \in T^0_{(p,\,l)} \cap \Psi_\nu^{\mathrm{irr}}$ where $(p, l_{\nu,j}) = (p, n \bmod \varphi(p^\nu))$. The root node (p,l) has height 0, so the exponent equals $1 + h^0_{(p,\,l)}(n)$.

If the tree $T^0_{(p,l)}$ has the height ≥ 1, then $(p, n \bmod \varphi(p^2)) \in T^0_{(p,l)}$; this implies that $h^0_{(p,l)}(n) \geq 1$. A trivial tree $T^0_{(p,l)}$ implies that $h^0_{(p,l)}(n) = 0$. Conversely, if $h^0_{(p,l)}(n) = 0$, then the height of the tree $T^0_{(p,l)}$ must be zero, otherwise we would get a contradiction. $\square$

Corollary 7.3 *Let n be an even positive integer, then*

$$\zeta(1-n) = (-1)^{\frac{n}{2}} \prod_{\substack{(p,l)\in\Psi_1^{\mathrm{irr}},\, \Delta_{(p,l)}\neq 0 \\ l\equiv n \,(\mathrm{mod}\ p-1)}} |p\,(\chi_{(p,l)} - \tfrac{n-l}{p-1})|_p^{-1} \prod_{\substack{(p,l)\in\Psi_1^{\mathrm{irr}},\, \Delta_{(p,l)}= 0 \\ l\equiv n \,(\mathrm{mod}\ p-1)}} p^{1+h^0_{(p,l)}(n)} \prod_{p-1|n} \frac{|n|_p}{p}$$

with $h^0_{(p,l)}$ as defined above.

Proof. This is a reformulation of Theorem 7.2 by $\zeta(1-n) = -B_n/n$ and (7.1). $\square$

8 An extension of Adams' theorem

Let n be an even positive integer. The trivial factor of B_n, given by (1.5), is a consequence of the implication, known as Adams' theorem, that

$$p^r \mid n \quad \text{with} \quad p-1 \nmid n \qquad \Longrightarrow \qquad p^r \mid B_n$$

where p is a prime and r is some positive integer. It was, however, never proved by Adams. In 1878 he computed a table of Bernoulli numbers B_{2m} for $m \leq 62$. On the basis of this table he conjectured† that $p \mid n$ implies $p \mid B_n$ for primes with $p-1 \nmid n$, see [1]. Note that the property that B_n/n is a p-integer for $p-1 \nmid n$ is needed to formulate the Kummer congruences (1.3). The case $r = 1$ of these congruences was proved by Kummer [15] earlier in 1851.

By Theorem 7.2 and the definitions of $h^0_{(p,l)}$ and $\chi_{(p,l)}$, we can state an extended version of Adams' theorem. We introduce the following notation. We write $p^r \mid\mid n$ when $p^r \mid n$ but $p^{r+1} \nmid n$, i.e., $r = \mathrm{ord}_p\, n$.

Theorem 8.1 *Let n be an even positive integer. Let p be a prime with $p^r \mid\mid n$, $r \geq 1$, and $p-1 \nmid n$. Let $l \equiv n \pmod{p-1}$ with $0 < l < p-1$. Then $p^{r+\delta} \mid\mid B_n$ with the following cases:*

(1) If p is regular, then $\delta = 0$,

(2) If p is irregular with $(p,l) \notin \Psi_1^{\mathrm{irr}}$, then $\delta = 0$,

(3) If p is irregular with $(p,l) \in \Psi_1^{\mathrm{irr}}$, $\Delta_{(p,l)} \neq 0$, then $\delta = 1 + \mathrm{ord}_p\,(\chi_{(p,l)} - \frac{n-l}{p-1})$,

(4) If p is irregular with $(p,l) \in \Psi_1^{\mathrm{irr}}$, $\Delta_{(p,l)} = 0$, then $\delta = 1 + h^0_{(p,l)}(n)$.

Additionally, in case (3) resp. (4), if $(p,l,l) \notin \widehat{\Psi}_2^{\mathrm{irr}}$, then $\delta = 1$, otherwise $\delta \geq 2$.

† "... I have also observed that if p be a prime factor of n which is not likewise a factor of the denominator of the nth number of Bernoulli, then the numerator of that number will be divisible by p. I have not succeeded, however, in obtaining a general proof of this proposition, though I have no doubt of its truth."

Proof. We have to consider the formula of Theorem 7.2. The first product yields $p^r \mid B_n$. Only the second resp. third product can give additional p-factors. Therefore case (1) and (2) are given by definition. We can now assume that $(p, l) \in \Psi_1^{\mathrm{irr}}$.

Case (3): A nonsingular $\Delta_{(p,\,l)}$ provides

$$\delta = \mathrm{ord}_p \, |p \, (\chi_{(p,\,l)} - \tfrac{n-l}{p-1})|_p^{-1} = 1 + \mathrm{ord}_p \, (\chi_{(p,\,l)} - \tfrac{n-l}{p-1}).$$

By assumption $n = p^r n'$ with some integer n'. We have to evaluate

$$d = \mathrm{ord}_p \, (\chi_{(p,\,l)} - \tfrac{n-l}{p-1}) = \mathrm{ord}_p \, (p \, \chi_{(p,\,l)} - \chi_{(p,\,l)} + l - p^r n').$$

Since $r \geq 1$, we p-adically obtain

$$(p, l, l) \in \widehat{\Psi}_2^{\mathrm{irr}} \iff \chi_{(p,\,l)} = l + s_3 \, p + \ldots \iff d \geq 1.$$

Conversely, $(p, l, l) \notin \widehat{\Psi}_2^{\mathrm{irr}}$ yields $d = 0$.

Case (4): A singular $\Delta_{(p,\,l)}$ provides $\delta = 1 + h^0_{(p,\,l)}(n)$. The definition of $T^0_{(p,\,l)}$ and Theorem 7.2 show that

$$(p, l, l) \notin \widehat{\Psi}_2^{\mathrm{irr}} \iff \text{the tree } T^0_{(p,\,l)} \text{ is trivial} \iff h^0_{(p,\,l)}(n) = 0.$$

Conversely, $(p, l, l) \in \widehat{\Psi}_2^{\mathrm{irr}}$ yields $h^0_{(p,\,l)}(n) \geq 1$. $\square$

So far, no $(p, l, l) \in \widehat{\Psi}_2^{\mathrm{irr}}$ has been found. The following corollary theoretically shows examples where δ is arbitrary large.

Corollary 8.2 *Assume that $(p, l, \ldots, l) \in \widehat{\Psi}_{r+1}^{\mathrm{irr}}$ exists with some $r \geq 1$. Set $n = lp^r$. Then we have $p^r \mid\mid n$ and $p^{2r+1} \mid B_n$, i.e., $\delta \geq r + 1$.*

Proof. By Definition 2.11 $(p, n) \in \Psi_{r+1}^{\mathrm{irr}}$ is associated with $(p, l, \ldots, l) \in \widehat{\Psi}_{r+1}^{\mathrm{irr}}$, since $n = lp^r = \sum_{\nu=1}^{r+1} l\varphi(p^{\nu-1})$. Thus, $p^{r+1} \mid B_n/n$ and finally $p^{2r+1} \mid B_n$. $\square$

Remark 8.3 As mentioned above, Johnson [10] calculated the now called irregular pairs $(p, s_1, s_2) \in \widehat{\Psi}_2^{\mathrm{irr}}$ of order two for $p < 8000$. He also proved that $(p, l, l) \notin \widehat{\Psi}_2^{\mathrm{irr}}$ resp. $s_1 \neq s_2$ in that range. In a similar manner, the nonexistence of irregular pairs $(p, l, l-1)$ of order two plays an important role in Iwasawa theory as seen in Section 6. One may conjecture that no such special irregular pairs (p, l, l) and $(p, l, l-1)$ of order two exist. But there is still a long way to prove such results, even to understand properly which role the zeros $\chi_{(p,\,l)}$ play. Now, we have the relation

$$(p, l, l) \notin \widehat{\Psi}_2^{\mathrm{irr}} \iff p^2 \nmid \widehat{B}(lp) \iff p^3 \nmid B_{lp}.$$

Yamaguchi [24] also verified by calculation that $p^3 \nmid B_{lp}$ for all irregular pairs (p, l) with $p < 5500$, noting that this was conjectured earlier by Morishima in general. The condition $p^3 \nmid B_{lp}$ is related to the second case of FLT, see [23, Thm. 9.4, p. 174]. See also [23, Cor. 8.23, p. 162] for a different context. Under the assumption of the conjecture of Kummer–Vandiver and that no $(p, l, l) \in \widehat{\Psi}_2^{\mathrm{irr}}$ exists, the second case of FLT is true for the exponent p. For details we refer to the references cited above.

The converse of Adams' theorem does not hold, but one can state a somewhat different result which deals with the common prime factors of numerators and denominators of Bernoulli numbers with indices close to each other.

Proposition 8.4 *Let $\mathcal{S} = \{2, 4, 6, 8, 10, 14\}$ be the set of all even indices m where the numerator of $|B_m/m|$ equals 1. Write $B_n = \mathit{\Lambda}_n/V_n$ with $(\mathit{\Lambda}_n, V_n) = 1$. Let k, n be even positive integers with $k \in \mathcal{S}$ and $n - k \geq 2$. Then*

$$D = (\mathit{\Lambda}_n, V_{n-k}) \quad \textit{implies} \quad D \mid n.$$

Moreover, if $D > 1$ then $D = p_1 \cdots p_r$ with some $r \geq 1$. The primes $p_1, \ldots, p_r$ are pairwise different and $p_\nu \nmid V_k$, $p_\nu \nmid B_n/n$ for $\nu = 1, \ldots, r$.

Proof. Assume that $D > 1$. We then have $D = p_1 \cdots p_r$ with some $r \geq 1$, since V_{n-k} is squarefree by (1.4). Let $\nu \in \{1, \ldots, r\}$. Since $p_\nu \mid \mathit{\Lambda}_n$ and $p_\nu \mid V_{n-k}$, we have $p_\nu - 1 \nmid n$ and $p_\nu - 1 \mid n - k$. From this we can deduce that $p_\nu - 1 \nmid k$ and consequently that $p_\nu \nmid V_k$. Next we assume that $p_\nu \nmid n$ or $p_\nu \mid B_n/n$. Note that $p_\nu \mid \mathit{\Lambda}_n$ and $p_\nu \nmid n$ imply that $p_\nu \mid B_n/n$, but not conversely. We can use the Kummer congruences (1.3) to obtain that

$$0 \equiv \frac{B_n}{n} \equiv \frac{B_k}{k} \pmod{p_\nu},$$

since $n \equiv k \pmod{p_\nu - 1}$. By the definition of the set $\mathcal{S}$ we have

$$\frac{B_k}{k} \not\equiv 0 \pmod{p_\nu} \tag{8.1}$$

which yields a contradiction. This shows that $p_\nu \mid n$ and $p_\nu \nmid B_n/n$. Finally it follows that $D \mid n$. □

Now, the set $\mathcal{S}$ cannot be enlarged, because (8.1) does not hold in general for numerators having prime factors. For example, let $p = 691$ and $n = 12 + (p - 1) = 702$, then we have $p \mid B_{12}/12$ and $D = (\mathit{\Lambda}_n, V_{n-12}) = pc \nmid n$ with some $c \geq 1$. On the other hand, one trivially obtains for $k \in \mathcal{S}$, p prime with $p - 1 \nmid k$, and $n = kp$ infinitely many examples of $D > 1$. In the following proposition, Proposition 8.4 plays a crucial role. Recall the definition of $S_n(m)$.

Proposition 8.5 *Let n, m be positive integers with even n. For $r = 1$, 2 we have*

$$m^{r+1} \mid S_n(m) \quad \iff \quad m^r \mid B_n.$$

Proof. We can assume that $m > 1$, since $m = 1$ is trivial. The case $n = 2$ follows by $B_2 = \frac{1}{6}$ and that $m^2 \nmid \frac{1}{6}m(m-1)(2m-1) = S_2(m)$ for $m > 1$. For now we assume that $n \geq 4$. We have, see (5.8), that

$$S_n(m) = B_n\, m + \binom{n}{2} B_{n-2} \frac{m^3}{3} + \sum_{k=3}^{n} \binom{n}{k} B_{n-k} \frac{m^{k+1}}{k+1}. \tag{8.2}$$

By (1.4) and the cases $B_0 = 1$ and $B_1 = -\frac{1}{2}$ the denominator of all nonzero Bernoulli numbers is squarefree. For each prime power factor $p^s \parallel m$ and k where $B_{n-k} \neq 0$ $(2 \leq k \leq n)$ we have

$$\operatorname{ord}_p \left(\binom{n}{k} B_{n-k} \frac{m^{k+1}}{k+1} \right) \geq s(k+1) - 1 - \operatorname{ord}_p(k+1) \geq \lambda\, s \tag{8.3}$$

with the cases: (1) $\lambda = 1$ for $k \geq 2$, $p \geq 2$, (2) $\lambda = 2$ for $k \geq 2$, $p \geq 5$, and (3) $\lambda = 3$ for $k \geq 4$, $p \geq 5$. The critical cases to consider are $p = 2, 3, 5$ and $s = 1$. Now, we are ready to evaluate (8.2) $(\bmod\ m^r)$ for $r = 1, 2$. Write $B_n = \mathit{\Lambda}_n / V_n$ with $(\mathit{\Lambda}_n, V_n) = 1$.

Case $r = 1$: Assume that $(m, V_n) > 1$. By (8.3) (case $k \geq 2$, $p \geq 2$) we obtain

$$S_n(m) \equiv B_n\, m \equiv \frac{\mathit{\Lambda}_n}{V_n}\, m \not\equiv 0 \pmod{m}.$$

Therefore, $(m, V_n) = 1$ must hold which implies $2 \nmid m$, $3 \nmid m$, and $p \geq 5$. Hence, by (8.3) (case $k \geq 2$, $p \geq 5$), we can write $S_n(m) \equiv B_n\, m \pmod{m^2}$. This yields

$$m^2 \mid S_n(m) \iff m \mid B_n. \tag{8.4}$$

Case $r = 2$: We have $m \mid B_n$ and $(m, 6) = 1$, because either $m^2 \mid B_n$ or $m^3 \mid S_n(m)$ is assumed. The latter case implies $m^2 \mid S_n(m)$ and therefore $m \mid B_n$ by (8.4). Since $|\mathit{\Lambda}_4| = 1$, we can assume that $n \geq 6$. We then have $B_{n-3} = 0$ and we can apply (8.3) (case $k \geq 4$, $p \geq 5$) to obtain

$$S_n(m) \equiv B_n\, m + \frac{n(n-1)\mathit{\Lambda}_{n-2}}{6 V_{n-2}}\, m^3 \pmod{m^3}. \tag{8.5}$$

Our goal is to show that the second term of the right side of (8.5) vanishes, but the denominator V_{n-2} could possibly remove prime factors from m. Proposition 8.4 asserts that $(\mathit{\Lambda}_n, V_{n-2}) \mid n$. We also have $(m, V_{n-2}) \mid n$ since $m \mid B_n$. This means that the factor n contains those primes which V_{n-2} possibly removes from m. Therefore the second term of (8.5) vanishes $(\bmod\ m^3)$. The rest follows by $S_n(m) \equiv B_n\, m \equiv 0 \pmod{m^3}$. $\square$

One cannot improve the value r in general. Choose $p = 37$ and $l = 37580$. Since $(p, l) \in \Psi_3^{\mathrm{irr}}$ we have $p^3 \mid B_l$, but $p^4 \nmid S_l(p)$ which was checked with **Mathematica**.

Example 8.6

(1) We have $B_{42} = 1520097643918070802691/1806$. Since the numerator $\mathit{\Lambda}_{42}$ is a large irregular prime, we obtain for $m > 1$ that

$$m^2 \mid S_{42}(m) \quad \iff \quad m = 1520097643918070802691.$$

(2) We have $\mathit{\Lambda}_{50} = 5^2 \cdot 417202699 \cdot 47464429777438199$ and $V_{48} = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17$. Hence, for $m > 1$ we have

$$m^3 \mid S_{50}(m) \quad \iff \quad m = 5.$$

A Calculations

Table A.1 B_n and B_n/n.

n	0	1	2	4	6	8	10	12	14	16	18	20
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$	$-\frac{3617}{510}$	$\frac{43867}{798}$	$-\frac{174611}{330}$
$\frac{B_n}{n}$			$\frac{1}{12}$	$-\frac{1}{120}$	$\frac{1}{252}$	$-\frac{1}{240}$	$\frac{1}{132}$	$-\frac{691}{32760}$	$\frac{1}{12}$	$-\frac{3617}{8160}$	$\frac{43867}{14364}$	$-\frac{174611}{6600}$

Table A.2 Calculated irregular pairs of order 100 of primes 37, 59, and 67.

Case $p = 37$.
Zeros of the sequence (s_ν) occur at index 19 and 81.

s_ν	1	2	3	4	5	6	7	8	9	10
0	32	7	28	21	30	4	17	26	13	32
10	35	27	36	32	10	21	9	11	0	1
20	13	6	8	10	11	10	11	32	13	30
30	10	6	8	2	12	1	8	2	5	3
40	10	19	8	4	7	19	27	33	29	29
50	11	2	23	8	34	5	8	35	35	13
60	31	29	6	7	22	13	29	7	15	22
70	20	19	29	2	14	2	2	31	11	4
80	0	27	8	10	23	17	35	15	32	22
90	14	7	18	8	3	27	35	33	31	6

Case $p = 59$.
Zeros of the sequence (s_ν) occur at index 31 and 95.

s_ν	1	2	3	4	5	6	7	8	9	10
0	44	15	25	40	36	18	11	17	28	58
10	9	51	13	25	41	44	17	43	35	21
20	10	21	38	9	12	40	43	45	30	41
30	0	3	25	34	49	45	9	19	48	57
40	11	13	29	28	44	41	37	33	29	43
50	8	57	12	48	15	15	53	57	16	51
60	16	54	30	9	26	8	49	22	58	11
70	42	28	36	33	45	24	32	18	12	29
80	45	40	27	19	40	41	11	42	49	35
90	41	57	54	33	0	34	34	49	6	31

Case $p = 67$.
Zeros of the sequence (s_ν) occur at index 23 and 85.

s_ν	1	2	3	4	5	6	7	8	9	10
0	58	49	34	42	42	39	3	62	57	19
10	62	10	36	14	53	57	16	60	22	41
20	21	25	0	56	21	24	52	33	28	51
30	34	60	8	47	39	42	33	14	66	50
40	48	45	28	61	50	27	8	30	59	32
50	15	3	1	54	12	30	20	14	12	10
60	49	33	49	54	13	26	42	8	58	12
70	63	19	16	48	15	2	13	1	23	2
80	44	64	25	40	0	16	58	44	31	62
90	47	61	46	9	2	50	1	62	34	31

Table A.3 Calculated irregular pairs of order 10 of primes below 1000.

(p,l)	$\Delta_{(p,l)}$	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
(37,32)	21	32	7	28	21	30	4	17	26	13	32
(59,44)	26	44	15	25	40	36	18	11	17	28	58
(67,58)	21	58	49	34	42	42	39	3	62	57	19
(101,68)	42	68	57	57	45	60	16	10	47	53	88
(103,24)	54	24	2	87	55	47	3	72	4	45	52
(131,22)	25	22	93	26	43	74	109	80	5	55	14
(149,130)	79	130	74	68	10	94	16	122	70	110	10
(157,62)	48	62	40	145	67	29	69	0	87	89	21
(157,110)	51	110	73	3	58	9	114	118	21	1	11
(233,84)	132	84	173	164	135	146	127	10	36	108	230
(257,164)	188	164	135	174	30	203	161	193	142	68	126
(263,100)	87	100	198	139	151	106	202	99	202	251	163
(271,84)	179	84	5	14	239	8	233	43	28	57	170
(283,20)	15	20	265	115	171	137	251	118	132	246	265
(293,156)	93	156	230	75	289	47	247	98	100	141	27
(307,88)	205	88	70	234	51	173	104	140	140	107	201
(311,292)	277	292	204	183	9	260	183	214	254	2	151
(347,280)	106	280	113	250	150	307	264	145	177	101	156
(353,186)	301	186	190	147	13	34	171	106	304	190	102
(353,300)	161	300	181	300	314	327	67	26	113	18	336
(379,100)	276	100	242	277	88	236	225	22	221	54	26
(379,174)	82	174	364	216	20	128	277	134	257	164	31
(389,200)	48	200	354	33	371	189	29	219	44	11	319
(401,382)	376	382	263	126	213	197	170	320	107	297	331
(409,126)	180	126	389	343	247	322	24	187	75	91	179
(421,240)	396	240	351	141	36	169	124	164	342	365	156
(433,366)	284	366	406	342	372	234	21	328	346	279	155
(461,196)	281	196	423	121	233	61	353	421	414	350	92
(463,130)	78	130	376	404	124	420	63	438	185	124	18
(467,94)	118	94	219	393	264	70	75	254	361	332	157
(467,194)	269	194	283	329	154	419	170	152	78	304	326
(491,292)	456	292	218	299	225	362	461	37	65	203	228
(491,336)	103	336	260	15	41	381	66	376	391	209	305
(491,338)	475	338	59	160	106	105	33	346	158	314	233
(523,400)	497	400	36	230	180	431	235	114	104	152	399
(541,86)	211	86	436	29	482	424	74	212	259	419	287
(547,270)	348	270	458	536	35	521	413	88	545	44	537
(547,486)	139	486	100	4	33	153	282	467	233	482	17
(557,222)	153	222	549	505	399	472	49	20	81	279	513
(577,52)	452	52	309	416	274	56	20	476	164	309	19
(587,90)	286	90	109	344	244	53	93	454	292	291	547
(587,92)	319	92	213	332	470	36	479	508	134	323	275
(593,22)	331	22	188	388	541	576	371	26	586	40	514
(607,592)	435	592	369	428	162	503	358	484	411	67	267
(613,522)	57	522	549	451	318	312	243	38	265	552	215
(617,20)	289	20	384	107	161	281	358	64	604	336	326
(617,174)	317	174	546	83	114	484	121	229	335	597	570
(617,338)	312	338	419	570	496	63	247	46	604	464	134
(619,428)	121	428	457	363	526	36	179	79	170	485	47
(631,80)	139	80	146	468	175	34	249	169	26	498	528
(631,226)	221	226	338	510	318	581	572	363	422	111	405
(647,236)	318	236	480	525	205	103	205	620	394	553	25
(647,242)	94	242	487	519	49	109	373	451	586	250	57
(647,554)	209	554	558	568	174	579	545	5	377	242	81
(653,48)	363	48	154	558	439	300	59	541	242	205	47
(659,224)	200	224	140	131	396	158	367	79	256	620	615
(673,408)	325	408	26	64	257	158	213	430	659	144	600
(673,502)	585	502	293	198	436	506	441	27	89	416	407

(p, l)	$\Delta_{(p,l)}$	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
(677,628)	440	628	504	457	324	461	88	532	653	89	244
(683,32)	477	32	266	20	625	119	190	13	190	222	214
(691,12)	611	12	496	104	197	607	590	303	96	461	152
(691,200)	592	200	496	333	578	93	160	436	611	215	278
(727,378)	398	378	683	722	169	391	150	694	210	228	130
(751,290)	164	290	481	37	181	27	31	71	8	36	164
(757,514)	554	514	364	164	375	7	720	750	273	592	643
(761,260)	462	260	729	680	274	188	464	183	283	52	235
(773,732)	517	732	147	306	278	370	412	89	340	637	223
(797,220)	375	220	369	279	501	300	168	530	534	747	268
(809,330)	88	330	52	743	100	336	157	759	348	43	736
(809,628)	18	628	773	629	623	160	494	339	244	463	274
(811,544)	381	544	424	100	346	749	624	220	410	313	62
(821,744)	704	744	621	319	498	427	50	21	237	305	809
(827,102)	105	102	164	443	469	568	671	183	372	512	464
(839,66)	269	66	135	305	36	40	659	431	326	591	293
(877,868)	480	868	554	279	714	821	520	76	565	104	22
(881,162)	789	162	372	330	89	244	27	229	418	438	89
(887,418)	611	418	76	698	835	872	130	319	217	439	573
(929,520)	607	520	433	27	711	366	902	838	7	351	805
(929,820)	706	820	749	156	59	913	480	432	114	129	491
(953,156)	24	156	720	516	620	229	251	77	805	689	477
(971,166)	715	166	538	594	897	509	355	749	180	174	96

Table A.4 Calculation: $n = 1$, $p = 37$, $l = 32$, $(p, l) \in \Psi_1^{\mathrm{irr}}$.

j	Index	$\alpha_j \pmod{p^3}$	$\equiv \pmod{p^3}$	$\Delta_{\alpha_j} \pmod{p^3}$	$\Delta_{\alpha_j} \pmod{p^2}$
0	32	3941/2720	42144	45827	650
1	68	2587/15	37318	49934	650
2	104	3821/1272	36599	30768	650
3	140	6497/7198	16714		$\Delta_{(p,l)} = 21$

Using Proposition 2.10 with $r = 3$ and $(r-1)n = 2$ yields $s \equiv 1043 \pmod{p^2}$ and $l_3 = 32 + s\varphi(p) = 37580$. We obtain $(37, 284) \in \Psi_2^{\mathrm{irr}}$, $(37, 37580) \in \Psi_3^{\mathrm{irr}}$, and $(37, 32, 7, 28) \in \widehat{\Psi}_3^{\mathrm{irr}}$.

Table A.5 Calculation: $n = 3$, $p = 37$, $l = 37580$, $(p, l) \in \Psi_3^{\mathrm{irr}}$.

j	Index	$\alpha_j \pmod{p^3}$	$\equiv \pmod{p^3}$	$\Delta_{\alpha_j} \pmod{p^3}$
0	37580	11241/22913	24645	45827
1	86864	49609/46188	19819	45827
2	136148	5261/24	14993	$\Delta_{(p,l)} = 21$

Using Proposition 2.9 yields $s \equiv 6607 \pmod{p^3}$ and $l_6 = 37580 + s\varphi(p^3) = 325656968$. We obtain $(37, 325656968) \in \Psi_6^{\mathrm{irr}}$, $(37, 55777784) \in \Psi_5^{\mathrm{irr}}$, $(37, 1072544) \in \Psi_4^{\mathrm{irr}}$, and $(37, 32, 7, 28, 21, 30, 4) \in \widehat{\Psi}_6^{\mathrm{irr}}$.

Table A.6 Calculation: $n = 3$, $p = 37$, $l = 37580$, $(p, l) \in \Psi_3^{\mathrm{irr}}$.

j	Index	$\alpha_j \pmod{p^9}$	$\equiv \pmod{p^9}$
0	37580	3791602112159/3307480	45520991695194
1	86864	1046892158059/484258896735	47985230204445
2	136148	13280633201029/15	70198303437443
3	185432	8822143378793/98280020	73479320052104

Using Proposition 5.1 with $r = 4$ and $(r-1)n = 9$ yields the sequence $21, 30, 4, \ldots, 27$ which provides $(37, 32, 7, 28, 21, 30, 4, 17, 26, 13, 32, 35, 27) \in \widehat{\Psi}_{12}^{\mathrm{irr}}$.

Note that Tables A.2 and A.3 were calculated with smallest possible indices of the Bernoulli numbers using Proposition 5.1; they agree with these results above. Additionally, the results were checked by Corollary 4.23 and Proposition 5.3. The program **calcbn** [11, Section 2.7] was used to calculate these large Bernoulli numbers extremely quickly.

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